THE SECOND REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE second regular meeting of the Southwestern Section of the Society was held at the University of Kansas, Lawrence, Kansas, on Saturday, November 28, 1908. About fifty persons, including the following seventeen members of the Society, were present:

Professor W. C. Brenke, Miss Julia T. Colpitts, Professor E. C. Colpitts, Professor E. W. Davis, Dr. Otto Dunkel, Professor E. P. R. Duval, Professor C. C. Engberg, Professor A. B. Frizell, Mr. G. W. Hartwell, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. W. A. Luby, Professor H. B. Newson, Dr. Mary W. Newson, Professor W. H. Roever, Professor H. E. Slaught, Professor Paul Wernicke.

The morning session was opened at 10 A. M. and the afternoon session at 2 P. M., Professor Newson presiding. The reading of papers was followed by a business meeting, at which Columbia, Mo., was fixed as the next place of meeting, and Professors Davis (chairman), Roever, and Kellogg (secretary) were elected as program committee for the ensuing year.

The following papers were presented:

(1) Professor J. N. VAN DER VRIES: "The steinerians of quartic surfaces."

(2) Professor G. A. MILLER: "On the groups generated by two operators satisfying the condition $s_1 s_2 = s_2^{-2} s_1^{-2}$."

(3) Professor E. W. DAVIS: "The imaginary elements of the exponential curve."

(4) Mr. MEYER GABA: "A necessary condition for the Cremona transformation of curves" (preliminary communication).

(5) Professor W. M. ROEVER: "Optical interpretation of some problems in statics."

(6) Professor W. C. BRENKE: "Transformation of series by means of functions admitting a recurrent relation."

(7) Professor A. B. FRIZELL: "Some sets whose cardinal is the second transfinite number."

(8) Professor PAUL WERNICKE: "Note on linkages."

(9) Professor A. B. FRIZELL: "A theorem on operation groups."

(10) Professor E. R. HEDRICK: "Integrals independent of the path in a complex field."

(11) Mr. G. W. HARTWELL: "Fields of force invariant under projective transformations."

(12) Professor PAUL WERNICKE: "Note on curvatures."

Professor Van der Vries and Mr. Gaba were introduced by Professor Newson. Professor Miller being absent, his paper was read by title. This paper appeared in the January BULLETIN. Abstracts of the other papers follow, numbered to correspond with the above list.

1. Professor Van der Vries considered the steinerian of the developable quartic both geometrically and analytically, and finds that it represents all of space. The quartic

$$ax^2s^2 + by^2z^2 + xyzs = 0$$

is examined analytically and the steinerian is found to be each of the four uninodal planes of the surface seven times and another quartic surface having all the singularities of the original quartic. The steinerian of the general quartic surface being of degree 32 is too difficult to admit of analytical methods. Surfaces having particular singularities are considered and the corresponding points of the steinerian located. The paper closes with a few general statements and proofs of properties of hessians and the corresponding properties of the steinerians.

3. Using the method of representing imaginaries which Professor Davis has previously presented to the Society, the periodicity of the exponential function as well as the peculiarity at infinity are interpreted geometrically.

4. Every Cremona transformation can be reduced to a series of quadratic Cremona transformations and collineations. A well-known necessary condition that a curve be transformable into another by a quadratic (and consequently by the general) Cremona transformation is that the genus of the two curves be the same. In Mr. Gaba's paper, a definition of complete rth adjoint, variable rth adjoint, and complement is given. A second necessary condition for the transformability of one curve into another is then derived., viz., that both curves have rth

variable adjoints of the same genus and the same degree of freedom, and that both curves lack the same variable sth adjoints.

5. Consider an apparatus consisting of a smooth wire and two small rings, P_1 and P_2, not on the wire, all rigidly attached to a frame. The wire is in the form of any curve, and on it a small weightless ring R is capable of sliding without friction. A weightless and perfectly flexible string with a weight at one end, after passing through the rings P_2 and R, has its other end attached to P_1. The string slides through the rings P_2 and R without friction. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the wire. If now the rings P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see images (*i. e.*, actual brilliant points *) of the light at certain points Q of the wire. The first theorem of Professor Roever's paper is that the points P and Q are identical.

Let us now think of a weightless and perfectly flexible string with both of its ends attached at fixed points P_1 and P_2. On the string a small *heavy* ring R is capable of motion without friction. Consider this apparatus as situated in any field of force for which a force function exists. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the field. If now the points P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see at P an image of the light (*i. e.*, an actual brilliant point) in the equipotential surface which passes through P. If no force function exists, the point P will be a virtual extra brilliant point of the line of force which passes through P. If in particular the points P_1 and P_2 coincide, the apparatus becomes a plumb line. Then the point P is the image which the observer sees of his eye in the equipotential surface which passes through P.

An apparatus was shown which illustrates the first theorem, and an apparatus was described which illustrates the second theorem. In the second apparatus the equipotential surface is actually assumed by the surface of water which reflects light.

6. In Professor Brenke's paper the series $\sum_1^\infty u_n \psi_n(x)$ is transformed by means of an assumed recurrent relation of the

* For the definitions of the different kinds of brilliant points, see *Transactions Amer. Math. Society*, vol. 9, No. 2, pp. 245–279.

form $\phi_r\psi_n = \alpha_n\psi_{n+r} + \beta_n\psi_{n+s}$, and a theorem on the convergence of the series is stated. Application is made to the cases where $\psi_n(x)$ is one of the functions $\cos nx$, Hermite's polynomial $U_n(x)$, Legendre's polynomial $X_n(x)$, and Bessel's function $J_n(x)$. By specializing the value of x, certain transformations and convergence tests are obtained for the series $\sum_1^\infty u_n$.

7. All permutations of a finite set of symbols may be obtained from any one of them by the enumerative process of filling the first place in all possible ways, then the second, and so on. The example of the continuum proves that this does not hold for infinite sets. Professor Frizell points out that a set whose cardinal is that of all permutations of the series 1 2 3 . . . can be obtained from this series by transposing pairs of consecutive numbers, *e. g.*, 21 43 56 . . . a property which obviously is not shared by finite sets. Transpositions introduced according to the method used by Hardy in building up a set whose cardinal is the second transfinite number $\aleph_1$ furnish an interesting set of sequences with the same cardinal.

8. If by means of non-intersecting lines (links) we connect $2v$ points (vertices) in 3-space, linking each of them to three others, we obtain a linkage with simple vertices. Such linkages are formed by the boundaries of certain maps. The proof of the map color theorem rests on showing that the links of a map's linkage may be marked by 3 indices α, β, γ so that no two bearing the same mark concur in a vertex. It is not possible, as Tait suggests, to apply such marking to the general simple vertex linkage, but it is shown in Professor Wernicke's paper that the conditions under which it can be done, cover the case of the "planar" linkage (that of the plane map).

9. Professor Frizell's second paper establishes the following proposition: Given two rules of combination of which the first is distributive relatively to the second, and a denumerable set of symbols forming a group with respect to the first rule, a semigroup with respect to the second, and a simply ordered class with reference to both, it is possible so to define extensions of the two given rules to the class of fundamental sequences of the given symbols that it shall possess the given properties with regard to the extended rules.

10. Independence of the path of integration is a characteristic property of analytic functions of a complex variable. Professor Hedrick first discusses integrals of the form $I_c = \int_c f'(z)dz$, where $f(z)$ is not analytic, and where $f'(z)$ is the value of the derivative in the direction of the path C. It results that I_c is independent of the path, that $\int_{z_0}^{z} f'(z)dz = f(z) - f(z_0)$, and that the values of $f'(z)$ on the elements of any radial field determine $f(z)$ except for a constant. The properties of $f'(z)$ on such a field are discussed.

In close connection with the preceding stands the discussion of integrals of the form $I_c = \int_c f(z)d\phi(z)$, where $f(z)$ and $\phi(z)$ are any functions of a complex variable. Such integrals, for real functions, have been considered by Kowalewski and by Ingold. Professor Hedrick shows that the integral I_c will be independent of the path, and that the formula for integration by parts holds if $f(z)$ and $\phi(z)$ together satisfy certain symmetrical conditions analogous to the Cauchy-Riemann equations, which turn out to be precisely the Beltrami forms for the surface element computed from $\phi(z)$.

11. A particle moving freely in a plane under the action of a force which depends only on the position of the particle has equations of motion of the form

$$md^2x/dt^2 = \phi(x, y), \quad md^2y/dt^2 = \psi(x, y).$$

The differential equation of the trajectories described by a particle under such conditions has the form

$$y''' = Hy'^2 + Gy'',$$

in which H and G are functions of y', ϕ, ψ, and the first partial derivatives of ϕ and ψ. This type of differential equation is invariant only under the group of collineations. In Mr. Hartwell's paper, the functions ϕ and ψ are found so that this differential equation will be invariant under the several groups of collineations classified by Lie. These functions do not exist in the case of groups of more than five parameters. The integral equations of the trajectories are readily obtained when the group under which the trajectories are invariant contains three or more parameters. The paper will appear in the *Transactions*.

12. For $n = 2$, 3, etc., the determinant

$$\begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 1 & x_1' & x_2' & \cdots & x_n' \\ 1 & x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix}$$

represents the $n!$-fold area, volume, etc., of the figure in n-space determined by $n+1$ points $(x_1, x_2 \cdots, x_n)$, $(x_1', x_2', \cdots, x_n')$, $\cdots$, $(x_1^{(n+1)}, x_2^{(n+1)}, \cdots, x_n^{(n+1)})$ and by the (higher) planes connecting every two, every three of them, etc. This figure possesses $\frac{1}{2}(n+1)\,n$ edges: $P_{01}, P_{02}, \cdots, P_{n,n+1}$, of lengths p_{01}, $p_{02} \cdots, p_{n,n+1}$, respectively; $(n+1)\,n\,(n-1)/3!$ 2-planes: P_{012}, $P_{013} \cdots, P_{n-1,n,n+1}$; $\cdots$ finally $n+1$ $(n-1)$-planes: $P_{0,1\ldots,n}$, $P_{0,1\ldots,n+1} \cdots, P_{1,2\ldots,n+1}$.

Another expression for the same quantity is obtained by repeated multiplication of "bases" with corresponding altitudes:

$$p_{01} \cdot p_{12} \sin(P_{01}, P_{12}) \cdot p_{23} \sin(P_{12}, P_{23}) \sin(P_{012}, P_{123})$$
$$p_{34} \sin(P_{23}, P_{34}) \sin(P_{123}, P_{234}) \sin(P_{0123}, P_{1234}) \cdots$$

Taking $n+1$ "consecutive" points on a general curve in n-space, and dividing each factor of this expression by the arc element ds, it becomes $c_1^{n-1} c_2^{n-2} \cdots c_{n-1}$, c_ν denoting the νth curvature, while the determinant, divided by $ds^{\frac{1}{2}(n+1)n}$, easily reduces to

$$\begin{vmatrix} x_1' & x_2' & \cdots & x_n' \\ x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & & \vdots \\ x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix},$$

where the upper indices now refer to differentiation with respect to s. Professor Wernicke thus gives a simple and general method for obtaining expressions for the various curvatures of curves in n-space.

O. D. KELLOGG,
Secretary of the Section.

REMARKS CONCERNING THE SECOND VARIATION FOR ISOPERIMETRIC PROBLEMS.

BY PROFESSOR OSKAR BOLZA.

In his lectures on the calculus of variations (Göttingen, 1904–05), Hilbert has given an elegant modification of Weierstrass's proof of Euler's rule for isoperimetric problems, which reduces the proof to the consideration of an ordinary extremum with a condition. The object of the present note is to show how the same method can also be applied to the *second variation*.

§ 1. *Hilbert's Proof of Euler's Rule.**

We consider the problem of minimizing the integral

$$(1) \qquad J = \int_{x_1}^{x_2} f(x, y, y')dx, \qquad y' = \frac{dy}{dx},$$

with respect to the totality of curves $y = y(x)$ of class C' which join two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, lie in a certain region $\mathfrak{R}$ of the x, y-plane, and furnish for the integral

$$(2) \qquad K = \int_{x_1}^{x_2} g(x, y, y')dx$$

a given value l. The functions f and g are supposed of class C'' in the domain

$$(x, y) \text{ in } \mathfrak{R}, \qquad -\infty < y' < +\infty.$$

Let

$$\mathfrak{E}_0: \qquad y = y(x), \quad x_1 \leqq x \leqq x_2$$

be a solution of the problem which lies in the interior of the region $\mathfrak{R}$, and which is not at the same time an extremal for the integral K. Then we can obtain, according to Weierstrass, a set of admissible variations of $\mathfrak{E}_0$ as follows:

Let $\eta_1(x)$, $\eta_2(x)$ be two arbitrary functions of x of class C' in the interval (x_1x_2), and vanishing at x_1 and x_2; then the curve

$$y = y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x) \equiv Y(x, \epsilon_1, \epsilon_2), \qquad x_1 \leqq x \leqq x_2$$

*The same proof is given by Kneser, "Euler und die Variationsrechnung," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Bd. XXV (1907), p. 50.

will be an admissible variation of $\mathfrak{C}_0$ provided that the two constants ϵ_1, ϵ_2 satisfy the relation

$$K(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} g(x, Y, Y')\,dx = l. \tag{3}$$

Hence,* *the function*

$$J(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} f(x, Y, Y')dx$$

of the two variables ϵ_1, ϵ_2 *must have, for* $\epsilon_1 = 0$, $\epsilon_2 = 0$, *a minimum restricted by condition* (3).

But if a function $f(x, y)$ has for $x = a$, $y = b$ a (relative) minimum restricted by the condition

$$\phi(x, y) = 0, \tag{4}$$

and if the two derivatives $\phi_x(a, b)$, $\phi_y(a, b)$ are not both zero, there exists a quantity λ such that

$$f_x(a, b) + \lambda\phi_x(a, b) = 0, \qquad f_y(a, b) + \lambda\phi_y(a, b) = 0, \tag{5}$$

and moreover, if we put

$$F = f + \lambda\phi,$$

the inequality

$$F_{xx}\phi_y^2 - 2F_{xy}\phi_x\phi_y + F_{yy}\phi_x^2\Big|_{\substack{x=a\\y=b}} \geqq 0 \tag{6}$$

must hold.†

Applying these results to the function $J(\epsilon_1, \epsilon_2)$, it follows in the first place that there must exist a constant λ such that the two equations

$$J_1 + \lambda K_1 = 0, \quad J_2 + \lambda K_2 = 0 \tag{7}$$

hold simultaneously, where

$$J_i = \left(\frac{\partial J}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (f_y\eta_i + f_{y'}\eta_i')dx,$$

$$K_i = \left(\frac{\partial K}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (g_y\eta_i + g_{y'}\eta_i')dx,$$

* Here Hilbert's proof branches off from Weierstrass's; the latter proceeds by solving (3) with respect to ϵ_2. Compare for instance my Lectures, p. 208.

†In order to prove these statements, it is only necessary to solve, by means of Dini's theorem on implicit functions, equation (4) with respect to y, $y = \mathfrak{y}(x)$, and to apply the ordinary rule for an extremum of functions of one variable to the compound function $f(x, \mathfrak{y}(x))$. The proof presupposes that f and ϕ are of class C'' in the vicinity of (a, b).

provided, however, that K_1 and K_2 are not both zero. But since $\mathfrak{C}_0$ is not an extremal for the integral K, we can always choose η_2 so that $K_2 \neq 0$.

The function η_2 being so chosen, the second of the equations (7) determines λ and shows that the value of λ is certainly independent of the choice of the function η_1. Hence it follows from the first of the equations (7), according to Du Bois-Reymond's lemma, that $y(x)$ *must satisfy the differential equation*

$$h_y - \frac{d}{dx} h_{y'} = 0, \tag{8}$$

where

$$h = f + \lambda g. \tag{9}$$

§ 2. *Application of Hilbert's Method to the Second Variation.*

We proceed next to apply the second necessary condition (6) to the extremum of the function $J(\epsilon_1, \epsilon_2)$. The condition takes here the form

$$K_2^2(J_{11} + \lambda K_{11}) - 2K_1K_2(J_{12} + \lambda K_{12}) + K_1^2(J_{22} + \lambda K_{22}) \geqq 0, \tag{10}$$

where

$$J_{ij} = \left(\frac{\partial^2 J}{\partial \epsilon_i \partial \epsilon_j}\right)_0, \qquad K_{ij} = \left(\frac{\partial^2 K}{\partial \epsilon_i \partial \epsilon_j}\right)_0.$$

But by an easy computation the inequality (10) reduces to

$$\int_{x_1}^{x_2} (h_{yy}\eta^2 + 2h_{yy'}\eta\eta' + h_{y'y'}\eta'^2)dx \geqq 0, \tag{11}$$

where

$$\eta = K_2\eta_1 - K_1\eta_2. \tag{12}$$

The inequality (11) must hold for all functions η_1, η_2 of class C' which vanish at x_1 and x_2, and it must be remembered that K_1 and K_2 vary with η_1 and η_2 respectively.

From the definition of K_1 and K_2 it follows that

$$\int_{x_1}^{x_2} (g_y\eta + g_{y'}\eta')dx = 0. \tag{13}$$

Conversely, the equation (12) represents the most general function η of class C' which vanishes at x_1 and x_2 and satisfies (13). For let η be any function satisfying these three condi-

tions. Then choose for η_2 any function which satisfies the two first of them and for which $K_2 \neq 0$. Let c be a constant arbitrarily chosen and determine η_1 by the equation

$$\eta = K_2\eta_1 - c\eta_2.$$

Then η_1 is of class C', vanishes at x_1 and x_2, and $c = K_1$, as follows from (13).

Hence we obtain the result: *The inequality* (11) *must be satisfied for all functions* η *of class* C' *which vanish at* x_1 *and* x_2 *and satisfy the relation* (13).

This is the theorem which we wished to establish by means of Hilbert's method. In order to recognize the advantages of the new proof, let us compare it with the ordinary proof which proceeds as follows:

Suppose we had found — no matter how — a one-parameter set of admissible variations of the curve $\mathfrak{C}_0$,

$$14) \qquad y = \bar{y}(x, \epsilon).$$

Then we must have for this set

$$(15) \qquad \delta^2 J \geqq 0.$$

In the discussion of this inequality *two difficulties* arise which have no analogue in the corresponding discussion for the unconditioned problem:

1) The integrand in $\delta^2 J$ contains *the second variations* $\delta^2 y$, $\delta^2 y'$, whereas in the unconditioned problem only the first variations δy, $\delta y'$ occur, owing to the fact that in the latter problem it is sufficient to consider variations of the simplest type

$$\bar{y}(x, \epsilon) = y(x) + \epsilon\eta(x),$$

which is no longer possible in the isoperimetric problem.

This difficulty is removed by eliminating* $\delta^2 y$, $\delta^2 y'$ by means of the equation $\delta^2 K = 0$, the result being

$$(1) \qquad \delta^2 J + \lambda\delta^2 K \geqq 0.$$

2) This inequality, which contains only the first variations δy, $\delta y'$, must hold for all functions δy which can be derived from an admissible variation (14) by the δ-process. But the totality of these functions is identical, according to a lemma† due

* Compare the remarks by Swift, BULLETIN, vol. 14 (1908), p. 373.

† Compare for instance my Lectures, p. 214.

to Weierstrass, with the totality of those functions δy of class C' which vanish at x_1 and x_2 and satisfy the relation $\delta K = 0$.

The proof of this lemma — which is an essential step in the chain of conclusions, and whose omission forms a serious gap in the older theory — constitutes the second difficulty.

Neither of these difficulties occurs in the proof which we have given above.

FREIBURG, i. B.,
November 19, 1908.

NOTES ON THE SIMPLEX THEORY OF NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, October 31, 1908.)

I. *Continued Product of the Terms of an Arithmetical Series.*

1. Let a and c be two relatively prime positive integers and form the arithmetical series

$$xa + c, \quad (x = 0, 1, 2, \cdots, n-1).$$

If we inquire what is the highest power of a prime p contained in the product

$$\prod_{x=0}^{x=n-1} (xa + c), \quad a \not\equiv 0 \pmod{p},$$

we shall find that the general result takes an interesting form. The solution of the problem may be effected in the following manner:

Evidently there exists some number x such that $xa + c$ is divisible by p. Let i be the smallest value of x for which this division is possible, and let c_1 be the quotient thus obtained. Using the notation

$$H\{y\} \tag{1}$$

to represent the index of the highest power of p contained in y, we will show that

$$H\left\{\prod_{x=0}^{x=n-1} (xa + c)\right\} = H\left\{\prod_{x=0}^{x=c_1} (xa + c_1)\right\} + c_1 + 1, \tag{2}$$

where

$$e_1 = \left[\frac{n-1-i_1}{p}\right]$$

is the largest integer not greater than $(n-1-i_1)/p$. In order to prove (2) we have only to notice that in the product of its first member only factors of the form

$$(mp + i_1)a + c$$

contain p and that the quotient of the division is always of the form

$$ma + c_1,$$

and that e_1 is the highest possible value of m. Performing the same operation on the H-function of the second member and continuing the process, we should finally arrive at a number which is simply the index of the required power of p.

In order to write this result in a convenient form let us define a suitable notation. Let i_r be the least integer such that $i_r a + c_{r-1}$ contains p and let c_r be the quotient of this division. For uniformity, set $c = c_0$ and $n - 1 = e_0$. Further, let e_r be defined by

$$\left[\frac{e_{r-1} - i_r}{p}\right] = e_r. \tag{3}$$

Also let t be the first subscript for which

$$c_t(a + c_t)(2a + c_t)\cdots(e_t a + c_t)$$

does not contain the factor p. Then the preceding result may be written thus

$$H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\} = \sum_{r=1}^{r=t-1}(e_r + 1). \tag{4}$$

Since $0 \leqq i_r \leqq p - 1$, as is evident from the definition of i_r, we may deduce from (3) the following inequalities:

$$\left[\frac{e_{r-1} - (p-1)}{p}\right] \leqq e_r \leqq \left[\frac{e_{r-1}}{p}\right].$$

Hence

$$\left[\frac{e_{r-1} + 1}{p}\right] \leqq e_r + 1 \leqq \left[\frac{e_{r-1} + p}{p}\right]. \tag{5}$$

This gives

$$\left[\frac{n}{p}\right] \leqq e_1 + 1 \leqq \left[\frac{n-1}{p}\right] + 1,$$

$$\left[\frac{n}{p^2}\right] \leqq e_2 + 1 \leqq \left[\frac{n-1}{p^2}\right] + 1,$$

$$\left[\frac{n}{p^3}\right] \leqq e_3 + 1 \leqq \left[\frac{n-1}{p^3}\right] + 1,$$

. . . . . .

Taking the sum of these inequalities, we have by (4)

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \leqq H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\}$$

(6)

$$\leqq \left[\frac{n-1}{p}\right] + \left[\frac{n-1}{p^2}\right] + \cdots + R(n-1),$$

where $R(n-1)$ is the index of the highest power of p not greater than $n-1$.

This result takes different forms according as n is or is not a power of p. If n is a power of p, we have evidently

$$\left[\frac{n}{p^a}\right] = \left[\frac{n-1}{p^a}\right] + 1 \tag{7}$$

for every p^a equal to or less than n. Remembering that when $n = p^h$

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = \frac{p^h - 1}{p-1},$$

and using equation (7) in connection with inequality (6), we have

$$H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\} = \frac{n-1}{p-1}, \quad n = p^h. \tag{8}$$

When n is not a power of p, it is evident that

$$\left[\frac{n}{p^a}\right] = \left[\frac{n-1}{p^a}\right]. \tag{9}$$

Suppose now that

$$n = \delta_h p^h + \delta_{h-1} p^{h-1} + \cdots + \delta_1 p + \delta_0, \quad \delta_h \neq 0, \tag{10}$$

and at least one other δ is not zero. Employing (9) and the

well-known formula

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(\delta_h+\delta_{h-1}+\cdots+\delta_1+\delta_0)}{p-1},$$

we may write (6) as follows:

$$\frac{n-(\delta_h+\cdots+\delta_1+\delta_0)}{p-1}\leqq H\left\{\prod_{x=0}^{x=n-1}(xa+c_0)\right\} \leqq h+\frac{n-(\delta_h+\cdots+\delta_1+\delta_0)}{p-1}. \tag{11}$$

The inequalities in (11) confine the value of H in narrow limits which are easily calculated.

2. In the series $xa+c$, it may happen that the first x for which $xa+c$ is divisible by p will give c as the quotient of this division. Then in the preceding discussion all the c's are equal; and then also all the i's. Dropping subscripts from i and c and making repeated use of equation (3), we have

$$e_1=\left[\frac{n-1-i}{p}\right],$$

$$e_2=\left[\frac{e_1-i}{p}\right]=\left[\frac{e_1p-ip}{p^2}\right]=\left[\frac{n-1-i-ip}{p^2}\right],$$

$$e_3=\left[\frac{e_2-1}{p}\right]=\left[\frac{e_2p^2-ip^2}{p^3}\right]=\left[\frac{n-1-i-ip-ip^2}{p^3}\right],$$

. . . . . .

If we add one to each member of each of these equations and take the sum of the results; then further, if we replace the resulting first member by its value taken from (4), we have

$$H\left\{\prod_{x=0}^{x=n-1}(xa+c)\right\}=\left[\frac{n-1-i+p}{p}\right] +\left[\frac{n-1-i-ip+p^2}{p^2}\right]+\left[\frac{n-1-i-ip-ip^2+p^3}{p^3}\right]+\cdots. \tag{12}$$

3. If $a=c=1$, equation (12) takes a very simple form. For this case $i=p-1$. The result is the well-known theorem that the highest power of p contained in $n!$ is that whose index is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(s_h+\cdots+s_1+s_0)}{p-1},$$

where

$$n = \varepsilon_h p^h + \varepsilon_{h-1} p^{h-1} + \cdots + \varepsilon_1 p + \varepsilon_0.$$

4. If $a = 2$ and $c = 1$, equation (12) takes a special form of considerable interest. The terms of $xa + c$ are the natural odd numbers in order, and p is an odd prime. It is evident that $i = \frac{1}{2}(p-1)$. Therefore

$$\left[\frac{n-1-i-ip\cdots-ip^{\beta-1}+p^\beta}{p^\beta}\right]$$
$$=\left[\frac{2n-2-2i-2ip-\cdots-2ip^{\beta-1}+2p^\beta}{2p^\beta}\right]=\left[\frac{2n-1+p^\beta}{2p^\beta}\right].$$

Then (12) becomes

$$H\{1\cdot3\cdot5\cdots(2n-1)\} = \left[\frac{2n-1+p}{2p}\right] + \left[\frac{2n-1+p^2}{2p^2}\right] + \left[\frac{2n-1+p^3}{p^3}\right] + \cdots.$$

II. *An Extension of Fermat's Theorem.*

It will be shown that the congruence

$$x^{\phi(n)} \equiv 1 (\text{mod } n),$$

where $\phi(n)$ is Euler's ϕ-function of n, is still true when the modulus is a multiple of n formed in a definite way, x being prime to the new modulus.

It has been shown * that $\phi(z) = a$ has always more than one solution. If z_1 and z_2 are two roots of $\phi(z) = a$, then z_1 and z_2 must each have a factor not common to the two except when one is an odd number and the other is twice that odd number; and hence, except in this case, their lowest common multiple is greater than either of them. Now if $z_1, z_2, \cdots, z_i$ are all the roots of $\phi(z) = a$, we have by Fermat's theorem the congruences

$$x^a \equiv 1 (\text{mod } z_1),\ x^a \equiv 1 (\text{mod } z_2),\ \cdots,\ x^a \equiv 1 (\text{mod } z_i),$$

where in each case x is prime to the modulus involved. Now if L is the lowest common multiple of $z_1, z_2, \cdots, z_i$ and x is prime to L, we have

$$x^a \equiv 1 (\text{mod } L), \tag{1}$$

where L is greater than any number whose totient is a except

* Carmichael, BULLETIN, vol. 13, p. 241.

when the equation $\phi(z) = a$ has only the two solutions $z = L$, $z = \frac{1}{2}L$. Hence,

THEOREM. *Except when n and $\frac{1}{2}n$ are the only numbers whose totient is the same as that of n, the congruence $x^{\phi(n)} \equiv 1$ holds for a modulus which is some multiple of n.*

A working method for finding such a modulus is the following:

Set $\phi(n) = a$, for convenience. Separate a into its prime factors and find the highest power of each prime p contained in a such that $\phi(p^a)$ is equal to or is a factor of a. Suppose that the following primes are found: $p_1^{a_1}, p_2^{a_2}, \cdots, p_j^{a_j}$. Then write out all the divisors of a and take every prime q such that $q - 1$ is equal to any one of these divisors, but $q \neq$ any p; and say we have $q_1, q_2, \cdots, q_k$. Then set

$$M = p_1^{a_1} p_2^{a_2} \cdots p_j^{a_j} q_1 q_2 \cdots q_k. \tag{2}$$

Then evidently

$$X^a \equiv 1 (\text{mod } M), \tag{3}$$

when X is prime to M. (It should be noticed that M may be a multiple of L in congruence (1).)

As thus defined, M is a definite function of a; say $M = M(a)$. For every odd value of a, except $a = 1$, we have $M(a) = 1$, as the reader may readily verify. Some even values of a give also $M(a) = 1$. There follows a table giving the value of $M(a)$ for each a for which $M \neq 1$ up to $a = 150$.

a	$M(a)$	a	$M(a)$	a	$M(a)$
2	12	48	2 227 680	104	12 720
4	120	52	6 360	106	1 284
6	252	54	43 092	108	22 265 704 680
8	240	56	6 960	110	33 396
10	132	58	708	112	26 740 320
12	32 760	60	3 407 203 800	116	7 080
16	8 160	64	32 640	120	279 390 711 600
18	14 364	66	388 332	126	549 092 628
20	6 600	70	9 372	128	65 280
22	276	72	10 087 262 640	130	17 292
24	65 520	78	948	132	50 483 160
28	3 480	80	18 400 800	136	10 960
30	85 932	82	996	138	1 646 316
32	16 320	84	285 962 040	140	13 589 400
36	69 090 840	88	491 280	144	342 966 929 760
40	108 240	92	5 640	148	17 880
42	75 852	96	432 169 920	150	12 975 732
44	2 760	100	3 333 000		
46	564	102	25 956		

III. *The Solutions of* $\phi(z) = a$.

It is desirable to have a general method for finding all the solutions of

$$\phi(z) = a$$

for any given a. The method used in Note II for finding M in congruence (1) is suggestive, and we may formulate a rule thus:

Find M as in Note II. Evidently, *the solutions of $\phi(z) = a$ will all be factors of M. Then examine all the factors of M and retain each one whose totient is a.*

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THE SOLUTION OF BOUNDARY PROBLEMS OF LINEAR DIFFERENTIAL EQUATIONS OF ODD ORDER.

BY PROFESSOR W. D. A. WESTFALL.

E. SCHMIDT[1] has studied the set of linear integral equations with non-symmetric matrix

$$(1)\quad \phi_i(s) = \lambda_i \int_a^b K(s,t)\psi_i(t)dt, \qquad \psi_i(s) = \lambda_i \int_a^b K(t,s)\phi_i(t)dt,$$

and has shown that, if there can be found for a function $f(x)$ a continuous function $h(x)$, such that

$$(2)\qquad f(x) = \int_a^b K(x,t)\,h(t)dt,$$

then

$$(3)\qquad f(x) = \sum_i \frac{\phi_i(x)}{\lambda_i} \int_a^b h(t)\,\psi_i(t)dt,$$

where ϕ_i runs over a complete set of solutions of (1) which have been normalized and orthogonalized, *i. e.*,

$$(4)\qquad \int_a^b \phi_i\psi_j dx = \left\{ \begin{array}{ll} 1, & i = j, \\ 0, & i \neq j. \end{array} \right\}$$

---

[1] *Math. Annalen*, vol. 63, p. 459.

A connection can be established with the boundary problem of differential equations as follows:

If the linear differential expression of odd order $L(y)$ is equal to its adjoint with negative sign, it satisfies the relation

$$(5)\qquad \int_a^b [zL(y) + yL(z)]\,dx = [P(y,z)]_a^b + [p_0(zy^{(n-1)} + yz^{(n-1)})]_a^b$$

$$(6)\qquad L(y) = p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y',$$

provided the p, y, and z satisfy certain continuity restrictions. $P(y, z)$ contains no derivatives of y or z of order higher than the $(n-2)$nd.

It is known * that the Green's function of $L(y)$ is skew-symmetric,

$$(7)\qquad G(s,t) + G(t,s) = 0,$$

and that no characteristic solution of $L(y) + \lambda_i y = 0$ exists for a real value of λ. Here $p_0, p_1, \cdots, p_{n-1}$ are real functions of the real variable x.

Let $\lambda_k = l_k + in_k$ be a complex value of λ for which there exists the characteristic solution

$$u_k(x) = \phi_k(x) + i\psi_k(x)$$

of $L(y) + \lambda_k y = 0$. Then $\bar{u}_k = \phi_k(x) - i\psi_k(x)$ is a characteristic solution of $L(y) + \bar{\lambda}_k y = 0$, $\bar{\lambda}_k = l_k - in_k$. Substitute $u = u_k$, $v = \bar{u}_k$ in (5).

$$2l_k \int_a^b u_k(x)\bar{u}_k(x)dx = 0.$$

Hence $l_k = 0$. Since u_k is a solution of $L(y) + in_k y = 0$, we have

$$L(\phi_k \pm i\psi_k) + in_k(\phi_k + i\psi_k) = 0,$$

or

$$(8)\qquad L(\phi_k) - n_k\psi_k = 0, \qquad L(\psi_k) + n_k\phi_k = 0.$$

Since u_k is a characteristic solution, ϕ_k and ψ_k constitute a set of characteristic solutions of equations (8). The relation (5) shows us that these satisfy with the Green's function $G(x, t)$ the equations

$$\phi_k(x) = n_k \int_a^b G(t,x)\psi_k(t)dt, \quad \psi_k(x) = -n_k \int_a^b G(t,x)\phi_k(t)dt.$$

* The author's dissertation, Göttingen, 1905.

Or from (7)

$$(9)\quad \phi_k(x) = n_k \int_a^b G(t,x)\psi_k(t)dt, \quad \psi_k(x) = n_k \int_a^b G(x,t)\phi_k(t)dt.$$

Moreover the solutions of equations (9), considered as integral equations with the known matrix $G(x, t)$, give a set of characteristic solutions of (8). This establishes the relation between Schmidt's pair of integral equations and the linear differential equation.

If $f(x)$ is continuous with its first n derivatives and satisfies the boundary conditions satisfied by the Green's function, equation (2) is solved by differentiation

$$L(f) = -h(x).$$

Hence there are an infinite number of pairs of solutions of (8), and an infinite number of characteristic solutions of $L(y) + in_k y = 0$.

COLUMBIA, MO.,
November 30, 1908.

A CLASS OF FUNCTIONS HAVING A PECULIAR DISCONTINUITY.

BY PROFESSOR W. D. A. WESTFALL.

CONSIDER all functions discontinuous for all rational values of the independent variable, and continuous and equal to zero for all irrational values. They are of the form

$$f\left(\frac{p}{q}\right) \neq 0, \quad p \text{ and } q \text{ prime to each other,}$$

(1) $f(\alpha) = 0$, for α irrational, with the condition that

$$\operatorname*{Lim}_{q=\infty} f\left(\frac{p}{q}\right) = 0.$$

The following are examples of such functions:

$$(2) \qquad \phi_n\left(\frac{p}{q}\right)=\frac{1}{q^n}, \quad \phi_n(1)=1, \quad \phi_n(\alpha)=0,$$

$$(3)^* \qquad \phi\left(\frac{p}{q}\right)=\frac{1}{q!}, \quad \phi(1)=1, \quad \phi(\alpha)=0,$$

$$(4) \qquad \psi\left(\frac{p}{q}\right)=\frac{1}{q^q}, \quad \psi(1)=1, \quad \psi(\alpha)=0.$$

Liouville has shown † that if α is an algebraic irrationality of the nth order

$$\left|\frac{p}{q}-\alpha\right|>\frac{A}{q^n},$$

where A is independent of q. Hence ϕ_n has a zero derivative at all algebraic irrationalities of order less than n, and ϕ and ψ at all algebraic irrationalities.

It will now be shown that it is impossible for a function of type (1) to have a derivative for every irrational value of the independent variable. Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of rational numbers such that

$$|a_n-a_{n-1}|<\frac{1}{|f(a_n)|} \quad \text{and} \quad <\left|\frac{1}{q^q}\right|,$$

if $a_n=p/q$. Then this defines a transcendental number α at which the difference quotient of $f(x)$ taken over the sequence a_n is greater than $\frac{1}{2}$. Hence a derivative cannot exist, since it is evidently zero if it exists. The definition of this point shows that such a point exists in every interval.

The above holds equally well for a function discontinuous at all rational points, and continuous at all irrational points in such a way that it coincides in these points with a function having a derivative throughout.

COLUMBIA, MO.,
December 12, 1908.

* An example first used by Professor Osgood in his lectures.
† *Comptes Rendus* for 1844.

ON CERTAIN DETERMINANTS CONNECTED WITH A PROBLEM IN CELESTIAL MECHANICS.

BY MR. H. E. BUCHANAN.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1909.)

1. THE solutions of the n-body problem, in which all are in the same straight line and moving in circles, are defined by the the following set of equations: *

$$
\begin{aligned}
0 + \frac{m_2}{r_{12}^2} + \frac{m_3}{r_{13}^2} + \cdots + \frac{m_n}{r_{1n}^2} &= -\omega^2 x_1, \\
-\frac{m_1}{r_{12}^2} + 0 + \frac{m_3}{r_{23}^2} + \cdots + \frac{m_n}{r_{2n}^2} &= -\omega^2 x_2, \\
-\frac{m_1}{r_{13}^2} - \frac{m_n}{r_{23}^2} + 0 + \cdots + \frac{m_n}{r_{3n}^2} &= -\omega^2 x_3, \\
\vdots \qquad\qquad & \\
-\frac{m_1}{r_{1n}^2} - \frac{m_2}{r_{2n}^2} - \qquad \cdots + 0 &= -\omega^2 x_n,
\end{aligned}
\tag{1}
$$

where the m_i stand for the masses and the x_i are their respective distances from the center of gravity of the system.

In his lectures Professor Moulton has considered these equations in two ways:

(1) Given the masses to find the distances.

(2) Given the distances to find the masses.

This paper is concerned with the second problem and its purpose is to answer two questions which he raised.

2. In (2) there is an unique determination of the masses if J, the determinant of the left members, never vanishes. J is a skew symmetric determinant, whose properties are quite different according as its order is even or odd. When n is even it is a perfect square,† and it will be shown that in this case J

*See Moulton's Celestial Mechanics, p. 213, for the three-body problem. The notation is here chosen so that $x_1 < x_2 < \cdots < x^n$.

†Cayley, Coll. Papers, 1, p. 412.

never vanishes for any selection of r_{ij}. It is only necessary to consider the Pfaffian

$$J^1 = \begin{vmatrix} \frac{1}{r_{12}^2} & \frac{1}{r_{13}^2} & \frac{1}{r_{14}^2} & \cdots & \frac{1}{r_{1n}^2} \\ & \frac{1}{r_{23}^2} & \frac{1}{r_{24}^2} & \cdots & \frac{1}{r_{2n}^2} \\ & & \frac{1}{r_{34}^2} & \cdots & \frac{1}{r_{3n}^2} \\ & & & & \vdots \\ & & & & \frac{1}{r_{n-1,n}^2} \end{vmatrix}. \tag{2}$$

Let a definite choice be made for all the x_i from x_1 to x_{n-1} inclusive. Let x_n vary between the limits x_{n-1} and $+\infty$. x_n can be written $x_n = x_{n-1} + x$, where x varies from 0 to $+\infty$. The Pfaffian (2) then becomes a function of x whose properties we propose to study:

$$P(x) = \begin{vmatrix} \frac{1}{r_{12}^2} & \frac{1}{r_{13}^2} & \cdots & \frac{1}{(x_{n-1} + x - x_1)^2} \\ & \frac{1}{r_{23}^2} & \cdots & \frac{1}{(x_{n-1} + x - x_2)^2} \\ & & & \vdots \\ & & & \frac{1}{x^2} \end{vmatrix}.$$

One can verify that $P(x)$ has the following properties:

(a) The terms of the last column are arranged in increasing order, $1/x^2$ being the largest.

(b) Differentiation as to x serves only to multiply $P(x)$ by a constant factor and to increase the power of the denominators in the last column.

(c) $P(x)$, as well as all its derivatives of finite order, remains finite and continuous for $0 < x < \infty$.

(d) $P(x)$ and all its derivatives vanish for $x = \infty$.

By Rolle's theorem and (c), (d), a necessary condition that $P(x)$ vanish for $0 < x < \infty$ is that $P'(x)$ vanish in the same interval. A necessary condition that $P'(x)$ vanish is that $P''(x)$ vanish, and so on; a necessary condition that $P^{(n-1)}(x)$ vanish is

that $P^{(n)}(x)$ vanish. On the other hand if $P'(x)$ never vanishes then $P(x)$ does not, and if $P''(x)$ never vanishes $P'(x)$ does not, finally, if $P^{(n)}(x)$ never vanishes $P^{(n-1)}(x)$ does not. From the properties (a) and (b) it follows that for n sufficiently great and $0 < x < \infty$, $P^{(n)}(x)$ never is zero since the term coming from $1/x^2$ will be greater than the sum of all the other terms unless its coefficient vanishes. This coefficient is the Pfaffian of order $n-2$, and we can apply the same method to it as to $P(x)$, so that it does not vanish unless the minor of order $n-4$ vanishes, etc. But the minor of order 2 does not vanish, therefore by induction none of the minors vanishes and the theorem is proven.

3. When n is odd the determinant of the left members of (1) is identically zero.* Then we consider† the first $n-1$ equations. Transpose the terms containing m_n and solve for $m_1, \cdots, m_{n-1}$. This is uniquely possible by § 2. Substituting these solutions in the last equation and arranging, there results

$$F(x_1, \cdots, x_n)\omega^2 + \phi(x_1, \cdots, x_n)m_n = 0.$$

$\phi(x_1, \cdots, x_n)$ is the determinant of the left members and is identically zero since n is odd. In order that there be a solution at all the following condition must be satisfied

$$F(x_1, \cdots, x_n) = 0.$$

m_n can then be taken arbitrarily and the remaining m_j are uniquely determined. Not only so, but it will be shown that after m_n and all of the x's except one has been chosen arbitrarily then that one is uniquely determined by the equation

$$F(x_1, \cdots, x_n) = 0.$$

4. One easily finds

$$F = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ & \dfrac{1}{r_{12}^2} & \cdots & \dfrac{1}{r_{1n}^2} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^2} \end{vmatrix} = 0.$$

* Cayley, Coll. Papers, I, p. 412.
† According to Moulton's Lectures.

Choose $x_2, \cdots, x_n$ arbitrary constants, while x_1 varies between the limits $-\infty$ and x_2. It is easily seen that

$$F(-\infty) = \infty, \quad F(x_2) = \infty,$$

the sign being undetermined. Then if the dF/dx_1 never vanishes F must vanish once and but once in the interval $-\infty < x_1 < x_2$. x_1 occurs in the second frame line; therefore on differentiating we have

$$\frac{dF}{dx_1} = \begin{vmatrix} 1 & x_2 & \cdots & x_n \\ & \dfrac{2}{r_{12}^3} & \cdots & \dfrac{2}{r_{1n}^3} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^2} \end{vmatrix}.$$

Now make a definite choice for x_1, say $x_1^{(0)}$, such that

$$-\infty < x_1^{(0)} < x_2;$$

then consider the Pfaffian

$$P_1(\mu) = \begin{vmatrix} \dfrac{\frac{1}{2}r_{12}^3}{(\frac{1}{2}r_{12} - \mu)^3} & x_2 & \cdots & x_n \\ & \dfrac{2}{(r_{12} - \mu)^3} & \cdots & \dfrac{2}{(r_{1n} - \mu)^3} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^2} \end{vmatrix}.$$

Let μ vary from $-\infty$ to $\frac{1}{2}(x_2 - x_1^0) = \frac{1}{2}r_{12}$; then in this interval $P_1(\mu)$ has the properties:

(a) The denominators of the second frame line are arranged in increasing order, $\dfrac{\frac{1}{2}r_{12}^3}{(\frac{1}{2}r_{12} - \mu)^3}$ being the smallest.

(b) Differentiation as to μ multiplies P_1 by a constant factor and increases the power of these denominators.

(c) $$P_1(0) = \frac{dF}{dx_1}\Big|^{x=x_1^{(0)}},$$

and P_1 and all its derivatives vanish for $\mu = -\infty$ but remain finite and continuous for $-\infty < \mu < \frac{1}{2}r_{12}$; and finally $P_1(\frac{1}{2}r_{12}) = \infty$. Reasoning then as in §2 we conclude that $P_1(\mu)$ never vanishes for any value, $-\infty < \mu < \frac{1}{2}r_{12}$; in particular it does not vanish when μ has the value 0, that is, by c,

$$\left.\frac{dF}{dx_1}\right|^{x=x_1^{(0)}} \neq 0.$$

Since the above proof does not depend on the value assigned to x_1 it follows that dF/dx_1 is never zero for any $-\infty < x_1 < x_2$ and therefore F vanishes once and but once in that interval.

5. In order to complete the theorem one must show that when x_i varies continuously between x_{i-1} and x_{i+1} F vanishes uniquely. The course of reasoning is the same as that above and we have to show that dF/dx_i never vanishes for $r_{i-1} < x_i < x_{i+1}$. x_i occurs in the ith frame line only. Again put $x_i = x_i^{(0)}$ a fixed constant such that $x_{i-1} < x_i^0 < x_{i+1}$ and consider the Pfaffian

$$P_i(\mu) = \begin{vmatrix} x_1 \cdots & q(\mu) & \cdots & x_n \\ \cdots & q(\mu)\dfrac{2}{r_{1i}^3} & \cdots & \dfrac{1}{r_{in}^2} \\ & \vdots & & \\ & q(\mu)\dfrac{2}{r_{i-1,i}^3} & \cdots & \dfrac{1}{r_{i-1,n}^2} \\ & \dfrac{2}{(r_{i,i+1}-\mu)^3} & \cdots & \dfrac{2}{(r_{in}-\mu)^3} \end{vmatrix}$$

$$q(\mu) = \tfrac{1}{2}(r^3_{i,i+1})/[\tfrac{1}{2}(r_{i,i+1}) - \mu]^3.$$

It can be shown that the coefficient of $q(\mu)$ in $P_i(\mu)$ does not vanish; then the properties (a) (b) (c) above hold, and the proof is completed as before.

CHICAGO, ILL.,
May, 1908.

SYLVESTER'S MATHEMATICAL PAPERS.

The Collected Mathematical Papers of James Joseph Sylvester. (Edited by H. F. BAKER.) Volume I, xii + 650 pp. Volume II, xvi + 731 pp. Cambridge, University Press, 1904, 1908.

PRIOR to 1877 there had been made in this country a few mathematical investigations of permanent value, in particular that by Benjamin Peirce on linear associative algebras. But such investigations had been done in isolation, and independently, in a measure, of contemporary European activity. These early writers were prophets, as it were, preparing the way for an era of wide-spread activity and interest in mathematical research. Chief among the forces which inaugurated this era was the advent of Sylvester at the opening of John Hopkins University in 1877, and his founding of the *American Journal of Mathematics* the following year. His high ideals for the search after truth, and his unbounded enthusiasm for mathematical science not only proved a powerful stimulus for his colleagues and students, but reached out beyond his immediate surroundings to the country at large.

Sylvester's work in this period, however, does not belong properly to the present review, since the first two volumes of his papers relate to the years 1837–1873. There occur 178 titles, including a few reports or abstracts of communications to societies. These latter titles appear to have been omitted from the Royal Society index, which shows 150 titles down to the year 1873. To judge from the later titles, including eighty-one for the next decade, there will probably be two additional volumes.

Sylvester's investigations related chiefly to algebra, including in that term the theory of invariants, matrices, theory of equations, etc., and the related domains of number theory, differential invariants, etc. However, there are various papers on astronomy, mechanics and physics, while a later volume will include his paper, "An application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics." His earliest paper related to Fresnel's optical theory of crystals; the next two and a paper of 1850 relate to motion and rest of fluids and rigid bodies. In

1856, there is "A trifle on projectiles" and a letter on Professor Galbraith's construction for the range of projectiles. In 1860 he wrote on the pressure of "mathematical" earth, the "essential quality which differentiates it from actual vulgar earth being continuity."

In the *Philosophical Magazine* for 1866, he has a twenty-four page article entitled "Astronomical prolusions; commencing with an instantaneous proof of Lambert's and Euler's theorems, and modulating through a construction of the orbit of a heavenly body from two heliocentric distances, the subtended chord, and the periodic time, and the focal theory of cartesian ovals, into a discussion of motion in a circle and its relation to planetary motion." After quoting from Lagrange: "His (Lambert's) theorem merits the especial notice of mathematicians, both on its own account, and because it appears difficult to arrive at by algebraical processes; so that it may be ranked among the small number of those in which geometrical seems to have the advantage over algebraical analysis," Sylvester makes some comments which might be called heresy on the part of the future Savilian professor of geometry at Oxford (in succession to the famous number-theorist Henry Smith), viz., "In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead over analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite."

In 1866 there is a sequel to the Astronomical prolusions with an equally ambitious title, and later on a supplement to the sequel. In the same year there is a rather extensive paper on the motion of a rigid body acted on by no external forces, and a shorter paper with a similar title. Soon afterwards, Sylvester became much interested in linkages and the general question of the conversion of motion. His lecture before the Royal Institution on January 23, 1874, "On recent discoveries in mechanical conversion of motion," brought to general notice the various possibilities of Peaucellier's linkage and inspired the well-known work of Hart and Kempe.

Sylvester's papers on geometry are not numerous. In 1850 there are elementary papers on the intersections and contacts of conics, and an "instantaneous demonstration of Pascal's

theorem." In 1851 there is an important paper on the "contacts of lines and surfaces of the second order," in which he discusses the reduction of a pair of quadratic forms U, V in two, three or four variables, notes explicitly the invariance of the highest common factor of the minors of order r of the determinant of $U + \lambda V$, and makes use of the multiplicity of the linear factors in classifying the forms. This work forecasts in a manner the more complete theory of Weierstrass of 1868. In 1854 there is a supplementary note giving certain corrections. In volume I, pages 378–381, is given Sylvester's paper of 1852, in which he enunciates his "Law of inertia for quadratic forms," which states the invariance of the number of positive coefficients in the form $\sum A_i \xi_i^2$ to which a real quadratic form may be reduced by a real linear transformation. His Probationary lecture on geometry, delivered in 1854, is given on the opening pages of volume II. On pages 236–244 are reproduced Sylvester's notes in the *Comptes Rendus* for 1861 on the involution of six lines in space, "double-sixes" of lines, and the construction of the 27 lines on a cubic surface. Two years later he considered questions of centers of gravity and the "principles of barycentric perspective."

Five notes or papers on the successive involutes to a circle appeared in 1868 and 1869. Adopting Cayley's name cyclode for the nth involute of a circle, Sylvester treated at length the reducible cyclodes in his paper in the *Proceedings of the London Mathematical Society* for 1869. He pursued this subject with the greatest enthusiasm. He remarks that "As crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich spiral has sprung the vast and interminable calculus of cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of denumeration, and by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the scriptural comparison of the kingdom of heaven to a grain of mustard-seed," etc. He concludes his article with a formidable list of prerequisites to the mastering of his exposition of the theory.

The theory of numbers proved attractive to Sylvester throughout his sixty years of scientific activity. His first notes related

to Wilson's theorem. In 1847 he discussed certain ternary cubic equations (recurring to the subject in 1879 and 1880). In the first of this series of papers, he took occasion to correct some errors in an earlier, hastily written, communication to the Institute of France. In the second of the series, he states that certain of his friends on the continent have "complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the theory of elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication." The third of the series is entitled "On the *general* solution (in certain cases) of the equation $x^3 + y^3 + Az^3 = Mxyz$." Sylvester begins as follows, "I shall restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify." In 1854 there was a further note on Wilson's theorem. In 1857 there were two papers on the partition of numbers, another on the resolvability of any integer into the sum of four squares, and a paper developing an idea of Eisenstein. The next year he wrote "On the Problem of the Virgins, and the general theory of compound partition," declaring himself at a loss to conjecture why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred by Euler to "the rule of the virgins," unless it has some mystical reference to the alligation of the coefficients of the two equations; he quotes from some correspondence with De Morgan on this point. In 1859, Sylvester delivered at King's College seven lectures on the partitions of numbers. Outlines of these lectures were circulated privately at the time, but were first published in 1897 without revision by the author in the *Proceedings of the London Mathematical Society*. These outlines appear in the present volume II, pages 119–175. In the example on 120 there seems to be no reason for the omission of the partition 3, 1, 1. Cayley and Sylvester appear to have become interested in the subject on account of its application to invariants; the subject had previously engaged the attention of Euler, Waring, De Morgan, Herschel, and others. Since Sylvester's

elaborate memoir on partitions in volume 5 of the *American Journal of Mathematics* belongs to a later period than the papers under review, the entire subject may appropriately be left to the care of the reviewer of the subsequent volumes. In 1860 and 1861 there were ten short papers on topics in the theory of numbers. In 1871 he discussed the Goldbach-Euler problem of the partition of an even number into two primes (recurring to the subject in later years), also some special cases of Dirichlet's theorem on the primes in an arithmetical progression.

The theory of substitution groups engaged Sylvester's attention in 1861. There were two notes in the *Comptes Rendus* relating to the number of substitutions of a certain kind. The note in the *Philosophical Magazine* related to an unsymmetrical six-valued function of six letters whose discovery had been ascribed by Cauchy to Hermite; Sylvester lays claim to priority in view of his paper of 1844 in the same journal, "On the principles of combinatorial aggregation."

Questions on tactic, a subject pursued nowadays in view of the applications in the algebraic theory of equations and in the foundations of geometry, were considered by Sylvester in the two papers last cited and three further papers of 1861 in the same journal. The results bear on substitution groups on nine letters and indirectly on the fifteen schoolgirl problem.

To the theory of determinants Sylvester made some well-known contributions which appeared in the *Philosophical Magazine* for 1851, and in the *Cambridge and Dublin Journal* for 1853. H. F. Baker, the editor of the present volumes, gives another exposition in present-day notations of Sylvester's main theorems on determinants (note to volume I).

Linear equations in finite differences were considered by Sylvester in 1862 in the *Comptes Rendus* and the *Philosophical Magazine*. In 1869 he gave in the latter journal "The story of an equation in differences of the second order."

Sylvester's dialytic method of elimination has passed into the very elements of mathematical instruction. It appeared in the *Philosophical Magazine*, 1840, pages 132, 379; 1842, page 534; 1851, page 221.

Sturm's functions were among the first objects of Sylvester's interest, and continued long to engage his attention. In 1841 he proceeded "to lay bare the internal anatomy of these remarkable forms." Divesting them of square factors, he obtains

expressions in terms of the roots,

$$f_2 = \Sigma(k-l)^2(x-a)(x-b)\cdots(x-h), \text{ etc.},$$

which Cayley characterized as "singularly elegant." In resuming the subject in 1853, he built up a more general theory which relates, not merely to $f(x)$ and $f'(x)$, but to any pair of polynomials $f(x)$ and $\phi(x)$. To these he applies Sturm's process of the greatest common measure and obtains functions which serve to indicate the manner in which the roots of $f(x) = 0$ are intercalated among the roots of $\phi(x) = 0$. This paper on the "syzygetic relations," etc., is perhaps Sylvester's most elaborate contribution; it concludes with a seven page "Glossary of new or unusual terms, or of terms used in a new or unusual sense, in the preceding memoir." Many of these terms are familiar to present students of invariant theory. Sylvester has said: "Perhaps I may without immodesty lay claim to the appellation of the mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all other mathematicians of the age combined." It has been remarked by Forsyth that Sylvester drew almost entirely upon Latin for new names, while Cayley as invariably drew upon Greek. To the same year, 1853, belong the two papers "On a remarkable modification of Sturm's theorem," *Philosophical Magazine*, where a certain series of quotients, arising from the continuedfraction development of f'/f is shown to be signaletically equivalent with Sturm's series of residues; also a paper in which these quotients are expressed in terms of the linear factors of $f(x)$. "Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much wished for representation, with which I propose now and for ever to take my farewell of this long and deeply excogitated theorem."

Newton's rule for the discovery of imaginary roots and an inferior limit to their number had remained undemonstrated, notwithstanding the attempts by Maclaurin, Campbell, Waring, Euler, Lagrange, and others. In 1864, Sylvester published a memoir of about a hundred pages in which he established the rule for equations of degree five or less, and gave "a complete invariantive determination of the character of the roots of the general equation of the fifth degree." The next year he announced a proof for the degrees six and seven of this

rule "so long the wonder and opprobrium of algebraists." In June of the same year 1865, Sylvester was rewarded by the discovery of a general proof and obtained a generalization of the rule. He contests with characteristic vigor the priority claim of J. R. Young. In 1866, Sylvester made a slight improvement in the statement of the generalized rule. The next year he presented "Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile work, and the theory of numbers." MacMahon states that one anallagmatic design by Sylvester was put down in the hall of the Junior United Service Club in Charles Street, Haymarket, but was unfortunately removed later whilst the hall was undergoing repair.

Reference should also be made to the elementary proof of Newton's rule which Sylvester gave in 1871. The theory of multiple roots had been investigated by Sylvester in 1841 and 1852; his theory for the limits to the real roots, given in 1853, connected with his investigation of Sturm's functions.

Canonical expressions for binary forms of even and odd degrees were first given by Sylvester in 1851 in three papers, of which the second was printed privately. To him the canonical form is "the very palpitating heart of the function laid bare to inspection." Cayley had previously reduced the general quartic to $x^4 + 6mx^2y^2 + y^4$. Employing the latter form, Sylvester proved in 1853 that every invariant of the general binary quartic is a rational integral function of the two well-known invariants of degrees 2 and 3, and similarly that Aronhold's S and T form a fundamental system ("scale," according to Sylvester) of invariants of the ternary cubic form. The subject of canonical forms has since been extended by Gundelfinger, Hilbert, and others.

The name invariant was introduced by Sylvester in 1851, *Philosophical Magazine*, II, page 396. Cayley's term had been hyperdeterminant.

Following the discovery of special invariants and covariants by Boole and Eisenstein, Cayley undertook a systematic study of invariants in 1845, and discovered processes for building invariants. In 1850 Hermite introduced his formes adjointes. These were obtained independently and from a simpler standpoint by Sylvester in 1851 in his brief paper, "On the general theory of associated algebraical forms"; he employs the term

contravariants. In the series of papers in the *Cambridge and Dublin Mathematical Journal*, 1852–1854, entitled "Principles of the calculus of forms," Sylvester carried the theory of invariants far beyond its previous stage, introducing concepts and and processes of first importance. Special mention must be made of his general processes for the construction of contravariants and concomitants, later designated as Überschiebungen by Gordan and made by him the foundation of the theory. Sylvester introduced and developed the class of concomitants of a system of forms known as combinants. In connection with Cayley, he developed the theory of commutants. Of the very few misprints noted, we may cite that for $1 \cdot 2 \cdot \cdots \cdot n$ on pages 305 and 306 of volume I.

Sylvester's extensive papers on reciprocants belong to a period subsequent to that covered by the present two volumes.

The tentative classification of the papers under review according to the topics applied mathematics, geometry, theory of numbers, substitution groups, tactic, determinants, finite differences, elimination, Sturm's functions, Newton's rule for imaginary roots, canonical forms and the theory of invariants, was employed by the reviewer to secure continuity of thought, even at the expense of historical sequence, and with the hope of calling the attention of specialists to Sylvester's papers in their fields. For the latter reason, mention has been made of many of his minor papers. Sylvester's most important contributions related to the theory of equations and invariants. Since his work in these subjects is so well-known, it was deemed unnecessary to go into details as fully as would be warranted, but quite sufficient to enumerate landmarks which would recall Sylvester's greatest achievements.

L. E. Dickson.

HILTON'S FINITE GROUPS.

An Introduction to the Theory of Groups of Finite Order. By Harold Hilton, M.A. Oxford, Clarendon Press, 1908. xii + 236 pp.

The theory of groups, which was first developed because of its relation to the solution of algebraic equations, today enters to a greater or less extent into the structure of many other de-

partments of mathematical thought, notably the theories of numbers, automorphic functions, algebraic substitutions, geometric transformations, and differential equations. Accordingly there is need of textbooks which will present the theory of abstract groups, not merely from the standpoint of permutation groups, but with a view to its many-sided applications to other concrete groups as well.

The book under review is evidently an attempt — and a very successful one it is — to meet this need, so far as groups of finite order are concerned. Heretofore the student has been obliged to depend, for the most part, either on books in which only abstract groups and permutation groups were considered, or on those in which groups were a secondary consideration in comparison with their applications and were therefore inadequately treated. In accordance with the well-known pedagogical principle that the concrete should always precede the abstract, it would seem that the theory of abstract groups should be copiously illustrated by examples chosen from the various kinds of concrete groups.

Moreover, for the beginner there is available plenty of illustrative material that is extremely elementary and easily comprehended. For instance, the additive and multiplicative groups of ordinary numbers, including the case where they are roots of unity; groups of translations; groups of rotations ot solids like the cylinder and the regular prism ; groups generated by reflections in the coordinate planes of a rectangular set ; the complement and supplement group in trigonometry ; the cross-ratio group of order 6 ; and the groups of classes of congruent integers with respect to a prime modulus, under addition and multiplication, respectively. All of these and many more like them are employed to advantage in the present text. Six of the earlier chapters are devoted entirely to concrete groups of three general kinds, permutation groups, substitution groups (chiefly groups of linear substitutions), and groups of movements. Throughout the remainder of the volume the contents of these chapters are constantly drawn upon for examples with which to illustrate and verify the general theorems.

In the development of the abstract theory a very careful and logical sequence of theorems is employed, long and intricate proofs are avoided, progress is made by short steps, and no step is taken before it is needed. As an instance in point the group concept itself is not introduced until the ground has been cleared

for it in the first four chapters. At the end of each step the author takes a fresh look around at the mathematical landscape opening before him; thus very little of the scenery escapes him, so far as he travels. The only omission of interesting elementary material that occurs to me is that of the determination of the distinct abstract groups of low order, as given, for instance, by Levavasseur in his Enumérations.

The reader will be struck by the condensed style and compact notation used in the book and by the unusual arrangement of its contents, at least two-thirds of which, I should judge, is in the form of examples; and forty-three pages at the end are devoted to "hints for the solution of the examples." Some of these examples are simple illustrations of the foregoing theory or easy steps in advance, but others are more difficult and would seem to require, from the beginner's standpoint, more extensive hints for their solution than those given. Indeed, to judge by some of the examples, one might almost suppose that the book was intended to be neither a textbook for beginners nor a treatise, but rather an encyclopedic compendium of theorems without proofs. But this is by no means the impression produced by the volume as a whole. Many of the examples are referred to in the proofs of later theorems and are, therefore, necessary links in the chain of reasoning. In short, the book is an extreme example of the use of the Socratic method in the writing of textbooks, but nevertheless, in the hands of an energetic student it will certainly prove valuable and stimulating.

The reader will note that the chief properties of the elements which go to make up groups are studied in detail before their combination into groups is introduced. The term element, as defined on page 1 is, therefore, restricted to those mathematical entities which are capable of combining into groups. For instance, not all singular matrices (those whose determinants vanish) are elements in this sense.

A few elementary theorems are given on infinite groups and semigroups. In regard to the latter it does not seem to have been noticed that if A is an element of infinite order, then according to any of the definitions given by De Séguier, Dickson, or Hilton, respectively, a semigroup can be formed by selecting powers of A in very unusual and peculiar ways, for instance, by selecting the 10th, 15th, 20th, 21st, and all higher powers of A. In order to avoid such oddities it would only be necessary to make an additional postulate to the effect that if ν is the

greatest common divisor of λ and μ then a semigroup which contains A^λ and A^μ must also contain A^ν.

The very latest discoveries, so far as they relate to the more fundamental parts of the subject, are embodied in the book. It is significant that commutators, which were first applied to finite groups by Miller in 1896, are introduced on page 4 and freely used throughout the book. As an evidence of the increasing vogue of the Galois field, the reader will observe that it is defined in the third chapter and that its properties are constantly used for illustrative purposes in the later chapters; also that such an elementary thing as the field of classes of congruent integers with respect to a prime modulus is defined merely as a special case of a Galois field. In examples 15–18, page 30, and example 3, page 32, certain theorems are stated concerning equations and polynomials whose coefficients are evidently intended to be integral marks of a Galois field, whereas they are incorrectly stated to be any marks of the field.

There is another oversight in example 11, page 70; it is true, as stated there, that an infinite abelian group is the direct product of a subgroup H formed by the elements of finite order and a subgroup K whose elements (besides 1) are of infinite order, but K does not, as stated there, contain *all* the elements of infinite order. The same slip was made by De Séguier.

It is to be noted that the definition of independent elements given on page 55 is not the usual one. According to this definition two elements of order 4 in the quaternion group, which are not inverse to each other, are independent, whereas they are usually regarded as mutually dependent, because they have the same square.

The condensation of the author's style sometimes tends to produce obscurity. On page 90, in examples 7 and 8, the reader is evidently supposed to understand that transitive groups are referred to, because the general subject of the section is k-ply transitive groups; but the insertion of the adjective transitive would seem to be required in order to avoid ambiguity.

Throughout the literature of the theory of groups there is a very general lack of precision in the use of the term subgroup; sometimes it includes identity and the entire group and sometimes not. Thus apparent contradictions are apt to arise. A striking instance is to be found on page 139 of Hilton's book, where example 3 contains the theorem that the central and

commutant of any group are characteristic (subgroups), while the example just before it states that a simple group has no characteristic subgroups!

The group of automorphisms (or isomorphisms) of a given group is advantageously defined on pages 136–137 in a purely abstract manner and not by means of its representation as a permutation group, as is usually done. In the last chapter a very brief introduction to the theory of group characteristics is given, and the subject is approached from the proper direction, namely, that of the representation of an abstract group as an irreducible linear substitution group. This route leads directly to the heart of the matter and makes it easily accessible, as Schur has shown in his "Neue Begründung."

In nomenclature there are a number of improvements upon existing usage. Thus "prime power group" replaces "group whose order is a power of a prime"; "commutant," "central," "dicyclic," and "normaliser" are apparently borrowed from De Séguier; "permutation" replaces the commoner word "substitution," in order to avoid confusion with the more general operation denoted by the term substitution; "point group" is used to denote a group of geometric movements that leave a given point fixed, and not, as in some of the older books, an "equivalent system of points" (and therefore not a group at all) which are carried into one another by the movements of a group. Since the word "class" has been much overworked, it would seem that the author might profitably have followed his own suggestion and used "speciality" instead. In the use of the expression "normal subgroup" Weber's example is followed. The useful distinction recently made between isomorphism and automorphism (sometimes called holomorphism) is preserved; but it is not evident why an un-English coinage like "not-square" should be retained, when the well-recognized prefix "non" is so immediately available.

The following misprints were found: page 44, § 12, line 5, "the single point P' corresponds to P" for "P' corresponds to the single point P"; page 53, example 17, (ii), "some constant" for "constants"; page 64, last line, "g" for "h"; page 140, line 2, "$2n^2$" for "$2n^4$"; page 189, § 2, example 8, (i), "x" and "n" are interchanged; page 212, § 9, example 4, (ii), "$\Delta = 0$" for "$\Delta \neq 0$."

There is a good index and an appendix containing a list of problems still unsolved. The book can be heartily commended

for three principal reasons, its embodiment of the results of the latest research, its carefully arranged sequence of theorems, and the emphasis which it lays on applications and illustrations.

ARTHUR RANUM.

SHORTER NOTICES.

Histoire des Mathématiques. Par W. W. ROUSE BALL. Edition française revue et augmentée. Traduite sur la troisième édition anglaise par L. FREUND. Tome deuxième. Avec des additions de R. de Montessus. Paris, A. Hermann, 1907. vi + 271 pp. 8 francs.

IT was natural to hope that Lieutenant Freund would succeed better with his second volume of Ball than he did with the first,* since he received warnings enough at the hands of numerous reviewers. In a way the hope has been realized, although in the mere matter of translation and of accuracy there is, unfortunately, no improvement.

The valuable part of this second volume, as it appears in its French form, lies in the additions made hy M. R. de Montessus, 'docteur ès-sciences mathématiques, lauréat de l'Institut.' These are numerous, particularly with respect to French mathematics, and they materially increase the helpfulness of the work as a book of reference. The period treated, being that beginning with Newton and running to the close of the nineteenth century, is one of great interest from the standpoint of the Paris mathematicians, and it was natural that a Cambridge writer could hardly do full justice to the labors of writers like l'Hospital, Varignon, De Montmort, D'Alembert, Laplace, and several others whose works Mr. Ball has briefly described. Other continental authors that are hardly mentioned at all in the English edition have fairly adequate biographical notices in the translation, and on this account the additions of Dr. de Montessus are particularly helpful. Among the added names are Meusnier, Lhuillier, Agnesi, Lacroix, Malfatti, Delambre, Montucla, Laurent, Cournot, Genocchi, Betti, Puiseux, Bouquet, Codazzi, Faà di Bruno, Catalan, Brioschi, Casorati, Halphen, Wronski, Bertrand, Laguerre, Stieltjes, Clebsch, Dupin, Chasles, Bellavitis, Cremona, Beltrami, Poinsot, Tisserand, and several others, none of whom are treated, or who at most are merely men-

* See review in the BULLETIN, vol. 12, p. 309.

tioned, in the English edition. Curiously enough it required a French translation to mention Waring, who one time held the Lucasian professorship, and whose "identity" would seem to have merited the insertion of his name, at least, in such an English work.

Unfortunately for the best use of the book, there is no index, and the table of contents is not sufficient to make up for the omission. Of all works, one of this nature, to which a student must often refer for a single topic or name, should have a complete index covering both volumes.

It has been said that the translation is not much of an improvement upon that of the first volume. If any proof of carelessness is needed, a glance at the table of 117 errata discovered by the author himself should suffice. But he has by no means covered the list, for even the casual reader will find scores of others. For example, for logarihmotechnica (page 14) read logarithmotechnia, and on the same page for x^{m-n-1} read x^{mn-1} twice; for $1 - y'^2$ read $1 - y^2$ (page 28); in Amérian, an English word, drop the accent (page 53); for V^2_2 read V_2^2 (page 54); for $\sin x + i \cos x$ read $\cos x + i \sin x$ twice on page 72, thus correcting an error that still exists in the fourth English edition, which has appeared since the translation, and on the same page change the date of De Moivre's death to 1754; spell the French for Edinburgh uniformly, both Edimbourg and Edinbourg appearing on page 73, and elsewhere; change Simpson's birth year from 1610 to 1710 (page 77); for "Arithmetic of lines" read "Arithmetic of sines" (page 82); for Englisch read English (page 89); for 1771 read 1781 (page 122). These are only typical of a considerable number of similar errors that will strike the reader, and they go to show how carelessly the work has been performed. We should be thankful for the added matter, but we might reasonably have hoped for a translation with a minimum instead of a maximum number of errors.

DAVID EUGENE SMITH.

Geschichte der Mathematik. I. Theil. Von den ältesten Zeiten bis Cartesius. Von Dr. SIEGMUND GÜNTHER. Leipzig, G. J. Göschen'sche Verlagshandlung, 1908. 8vo. 56 figures. viii + 427 pp. 9 Marks.

THE great interest manifested of late in the history of mathematics, evidenced by the fourth volume of Cantor, the enlarged

form of the *Bibliotheca Mathematica*, the success of the *Abhandlungen*, the Encyklopädie, and the translations of Ball, assures a hearty welcome to a work like this of Professor Günther's. If the undertaking had been merely a rewriting of old material or a condensation of Cantor, it would not be without its value, but since it is the product of a man with the historical ability and the reputation of Dr. Günther it is certain to take rank as an authority in its field.

The work was undertaken originally in collaboration with Professor von Braunmühl, the plan being that the author of the present volume should carry the work to the middle of the seventeenth century, and von Braunmühl from there on. What the effect of the recent death of the latter may have on the success of the undertaking it is unpleasant to consider, but it is to be hoped that the manuscript of the second volume was completed before the world was deprived of the services of one whose history of trigonometry will remain a standard authority for a long time to come.

This first volume consists of twenty chapters, as follows: Number and measure as primitive concepts of mankind, Mathematics in Mesopotamia, Mathematics in Egypt, Mathematics in China and ancient India, The prealexandrian period in Greece Classical antiquity, Greek mathematics between Apollonius and Ptolemy, Roman mathematics, The decline of Greek mathematics, Byzantine mathematics, Mediaeval Hindu mathematics, The early Arab period, The later Arab period, The position of science in the Church and Court schools of the Christian middle ages, Leonardo of Pisa, Mathematical teaching and progress in the later middle ages, The reform period of Peurbach and Regiomontanus and their followers to 1500, General characteristics of the 16th and early 17th centuries, The arithmetical sciences between 1500 and 1637, and The geometric and mechanical sciences in the same period. The work is followed by an index of names, the general index being presumably, and meantime unfortunately, left until the close of the second volume. There is also a helpful bibliography of seven pages, far from complete but valuable to beginners.

The first thought that may occur to a reader will relate to the reason for writing such a work, in view of the rather extensive literature that we have already upon the subject. The question is answered, however, as soon as he begins to examine the book. For one thing, mathematical history has advanced

since Cantor wrote his first two volumes, and new material is now available. This has been used sufficiently to be in itself a warrant for such a treatise. For example, Kugler's work of 1900 on the lunar tables of Babylon, Hilprecht's work on the Nippur excavations (1904, but not that of 1906), von Braunmühl's contributions, Tannery's latest articles — these and others of similar nature have contributed to make the book more up to date than any similar work available. Furthermore, greater seriousness of purpose and maturity of judgment characterize this history than are found in any other of the smaller treatises; that is to say, there is less of the anecdotal and more of the historical and mathematical than would appear in a work like Ball's, for example. On the other hand, the book has not the heaviness of Cantor, but preserves the readable style that is found in the author's other treatises.

In spite, however, of the fact that Professor Günther has given us the best elementary hand book of early mathematical history that has yet appeared, it is not without its points of weakness, partly from lack of sources, partly from errors of judgment. It seems unfortunate, for example, that one should attempt to write upon the history of Chinese mathematics without knowing more than the little given by Biernatzki, and by Biot in his translation of the Le Tcheōu Ly. To be sure the author mentions Wylie, but he has evidently never read his best contributions; while as to the works of Williams, Vissière, Knott, and Hyashi (and of course Endo's work in Japanese) he is apparently wholly ignorant. In the same line of bibliography one can readily understand that the author would be justified in omitting the older authorities if he thought best, but why he should mention Kaestner, for example, and omit Montucla, Libri, and Bossut, is not so easily explained. So in general, while the bibliography is helpful, it does not seem to be as complete as might reasonably be hoped, nor as well selected as it should be for its present extent. Had Vissière's excellent monograph been read, for instance, the too sweeping assertion about the antiquity of the swan pan (page 36) would not have been written, and had any serious study been made of the voluminous literature relating to the Yih-king more doubt would have been cast upon the Leibniz hypothesis with respect to the connection of the trigrams with a binary scale. The paucity of sources consulted is also manifest in the brief mention of Tschu schi kih (Chu Shi-ki) and his knowledge of

the binomial coefficients. Now it is true that the work of this author appeared "bald nach 1300," more exactly in 1303, and that it was quite epoch-making, but it is best known for the use made of several monads (unknowns) in systems of equations, and for the ingenious symbolism for polynomials. He was only one of three great writers, nearly contemporary, who created a renaissance of mathematics in China about that time, the other two being Tsin Kew-chaou, who introduced the monad as the symbol for the unknown and who did very creditable work on higher numerical equations, and Li-yay Jin-King, neither of whom are mentioned.

What is here said concerning Chinese mathematics is merely typical. It is easy to see that Professor Günther has used the best of the common secondary sources, but that rarely has he gone to those that may be called primary. Thus on the Greek abacus, he assumes that the tradition of its common use is to be accepted, when as a matter of fact it is extremely doubtful, and with respect to the similar question in Rome he asserts that we have no examples of such an instrument extant, while we have several that may quite possibly go back to classical times. As to the early printed arithmetics he cannot have consulted many original sources. For example, Licht's work was published without date, but probably about 1500, and not in 1514 as he asserts; Stromer's work appeared in 1504, and in two editions before the first one cited; the date of the Protomathesis of Finaeus is more properly 1530–32 than 1532; Ringelberg's Opera (not Chaos mathematicum) appeared at Leyden (not Lyons) in 1531; Ciruelo's Cursus quatuor appeared at Paris as well as at Alcalá in 1516; Ramus's 1569 work was really entitled Arithmeticæ libri dvo: Geometriæ septem et viginti, and reference should be made to his earlier work of 1555; the Opuscula mathematica of Maurolycus was published at Venice instead of Palermo, in 1575; the first edition of Moya was 1562 instead of 1609, which makes quite a difference in the argument on page 343; Nuñez's work appeared in 1564, the 1567 edition being the second. This list is closed simply because it would be unprofitable beyond this point, and not because it could not be greatly extended. It shows that Professor Günther has been unfortunate in his secondary sources or careless in his use of primary ones.

Independently of such errors, however, the fact remains that the book is the best that we have of its particular type.

DAVID EUGENE SMITH.

Manuscrits de Evariste Galois. Publiés par JULES TANNERY. Paris, Gauthier-Villars, 1908. 67 pp.

THE articles published by Galois related to periodic continued fractions, the algebraic solution of equations (summary of results), and the introduction of quantities (now known as Galois imaginaries) determined as the roots of irreducible congruences. These brief articles appeared during the years 1828–1830 in the *Annales de Mathématiques* de M. Gergonne, and the *Bulletin des Sciences mathématiques* de M. Férussac. They have been reprinted as pages 1–23 of Œuvres mathématiques d'Evariste Galois, Paris, 1897. Pages 25–61 of the latter book give a reproduction of certain posthumous papers by Galois, which first appeared in Liouville's *Journal de Mathématiques*, volume 11 (1846), pages 408–444, fourteen years after the tragic death of the author. These posthumous papers are the celebrated letter to Auguste Chevalier, his fundamental memoir on the conditions for the solution of algebraic equations by radicals, and an incomplete manuscript on solvable primitive equations.

Chevalier prepared for publication the manuscripts entrusted to him by Galois and placed them in the hands of Liouville. The latter published the three papers just mentioned, promising in a footnote to add later some other fragments extracted from the papers of Galois, which, although without great importance, could be read with interest by geometers. However, this additional manuscript by Galois is now published, for the first time, by Tannery, who gives in a most faithful manner a minute account of all the Galois manuscripts, points out as far as possible the few annotations by Chevalier, and indicates by detailed references the occasional divergence between the original manuscript by Galois, the copy by Chevalier, the text by Liouville, and that of the Œuvres. On page 15, line 7, the correct reading 1, 2, 3 should replace 1, 3, 3, and not 1, 1, 3.

In contrast to the statement in Picard's Introduction to the Œuvres, "les deux Mémoires qu'il présenta à l'Académie des Sciences ayant été perdus," Tannery has now made it clear, on pages 5 and 6, that the manuscript presented to the Academy by Galois was not lost, but is precisely the one which passed into the hands of Chevalier, thence to Liouville, and has finally been given to the Academy of Sciences by Liouville's daughter. What was rejected in 1831 has now been received with honor! The manuscript bears the names Lacroix and Poisson of the

members of the commission which passed upon this work of Galois on the solution of equations by radicals, now universally admitted to be one of the greatest triumphs of the human intellect. The verdict of the commission, however, delivered by Poisson in large handwriting, was simply "vu." The manuscript shows Galois's comment: "Oh! chérubins." There also appears in Liouville's writing "Rapport du 4 juillet 1831"; but he refrained from publishing the work until 1846 and then with all references to its earlier presentation to the Academy carefully deleted, though not on the first set of proofs still preserved. Liouville was the most conscientious of editors and there were details not clear to him in the manuscript of Galois; in fact, Liouville made various additions, afterwards suppressed, in an attempt to bring the manuscript into conformity with his precept "of being transcendentally clear." All honor to Liouville for recognizing the fundamental character of Galois's work and for preserving it to the profit of all ages.

The text of the manuscripts of Galois not previously published occupy over forty pages, including some work while still in school, and several introductions to memoirs merely planned, or else begun but left unfinished. The majority of the manuscripts are fragmentary notes bearing on the solution of equations by radicals. The second sentence on page 22 will be of interest. "Si maintenant vous me donnez une équation que vous aurez choisie à votre gré et que vous desiriez connaître si elle est ou non soluble par radicaux, je n'aurai rien à y faire que de vous indiquer le moyen de répondre à votre question, sans vouloir charger ni moi ni personne de le faire. En un mot les calculs sont impracticables." In defense of his method, however, he indicates that in practice one knows in advance properties of a proposed equation which will aid in applying his method. The same point may be raised in the application of Lie's theory. One does not make headway with equations, whether algebraic or differential, if selected at random or stripped of the information accompanying their origin.

M. Tannery has executed admirably the task he laid before himself. He has given to the world an impartial historical work of great interest, a necessary companion to the Galois text already available.

L. E. Dickson.

Diophantische Approximationen. Eine Einführung in die Zahlentheorie. Von HERMANN MINKOWSKI. Leipzig, Teubner, 1907. viii + 235 pp.

THE title of this original and highly interesting work is applied by the author to extensive classes of inequalities to be satisfied by integral values of the unknowns, the constants being any quantities; just as hitherto equations of the corresponding type have been designated with the name of Diophantus. In the first chapter, the author considers the rational approximation to one or more arbitrary real numbers, gives a new proof of the existence of integral solutions of $sx - ry = 1$, s and r being relatively prime integers, and establishes the theorem that there exist integral values, not all zero, of three variables such that each of the three given linear forms of determinant unity has a value not exceeding unity numerically. In the proof, due to a suggestion by Hilbert, the author introduces the concept of the minimum of a given system of linear forms for integral values of the variables and shows that the minimum changes continuously under continuous variation of the coefficients of the forms. An earlier proof by Minkowski, based upon geometric considerations, was given in his Geometrie der Zahlen, Leipzig, 1896.

Chapters 2 and 3, forming about half the book, relate to lattices; a Zahlengitter in two or three dimensions is defined to be the totality of sets of integral values of two or three variables, with the obvious geometric representation by a lattice. Here the author gives a detailed account of his theory of convex figures or solids having a center, in their relations to the lattice. In particular he discusses the following question: How may an infinitude of given congruent central convex solids be placed in parallel positions, such that no two of the solids overlap, such that the centers form a three dimensional lattice, and such that the space not filled by the solids shall be as small as possible? The problem is discussed at length and definitive analytic criteria are deduced (page 104). The text carries the discussion to a final conclusion for spheres and for ellipsoids and applies the results to the immediate derivation of a complete theory of positive ternary quadratic forms, obtaining in particular the results of Gauss and Seeber.

The second half of the text develops the more fundamental parts of the classic theory of algebraic numbers from the author's novel standpoint. The example of cubic number

fields $K(\theta)$ is usually chosen to secure simplicity in the geometric interpretations. The integral algebraic numbers of $K(\theta)$ correspond uniquely to the points of a lattice. The existence of a basis for the integral algebraic numbers follows at once geometrically. An inferior limit for the discriminant of a number field is established; except for the field of rational numbers the discriminant exceeds unity. Within the lattice which pictures the totality of integral algebraic numbers there exists (page 156) a lattice whose points picture uniquely the numbers of any given ideal. There follows geometrically the existence of a basis of the ideal. The introduction of lattices not only secures a highly serviceable geometric setting for the algebraic theory, but in connection with related concepts from the author's Geometry of Numbers enables him to establish some fundamental theorems on algebraic numbers which have not been demonstrated without this geometric theory. The final chapter on the approximation of complex quantities by means of numbers of the fields defined by a cube root or a fourth root of unity is the work of Dr. Axer, who also otherwise aided in the publication of the book.

Since the author has given an elementary, entertaining account, both in geometric and arithmetic language, of some important original results as well as the salient features of a classic theory, but presented in a novel manner, his work is deserving of the attention of the very widest circle of readers.

L. E. DICKSON.

Die Lehre von den geometrischen Verwandtschaften. Zweiter Band; die eindeutigen linearen Verwandtschaften zwischen Gebilden zweiter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1908. vi + 346 pp.

THE present volume of Professor Sturm's treatise has about the same programme for two dimensions as the preceding one has for one,* except that non-linear transformations are only considered incidentally. This part is not provided with an index nor a preface, but contains a glossary of 131 technical words besides those found in the preceding one, and the table of contents of all four volumes.

The first chapter (pages 1–218) is concerned with collineation and correlation in the plane, being somewhat similar in scope and

* See the review in the December number of the present volume of the BULLETIN, p. 135.

treatment to parts of Reye's treatise, volumes two and three. The proof that the vertices of any two quadrangles are sufficient to define projectivity in the plane is identical with that given by Cremona (Elements of projective geometry, Art. 94), but the author acknowledges assistance from a letter from Reye. After the general properties have been considered, metrical properties, particularly vanishing lines and foci are discussed; particular cases, such as affinity, similarity, and congruence are treated at length. In the discussion of two superposed collinear fields, the existence of one real self-corresponding point is assumed. The rest of the discussion is then clear enough, but it seems odd that no proof of this fundamental theorem is given. Homology is given an important place and Hermitian collineations are treated at length. Thus far correlation has hardly been mentioned; it is introduced by duality, showing that a pair of elements may be simply or doubly conjugate. If all elements are doubly conjugate, we have a polar field. The discussion of this part is very detailed; in connection with imaginary elements it is shown that loci exist, all of whose points are imaginary, but arranged in conjugate pairs (Klein's null-partite curves). A large number of constructions of polar fields is given, including the determination of foci, center, conjugate diameters of the invariant conic, and self-conjugate triangle of a pencil or a series of conics. These ideas are now developed in the geometry of the bundle, the procedure being quite similar to that given by Reye, volume 1, chapter 15, but followed by a large number of trigonometric formulas involving relations between conjugate diameters, cyclic planes and principal axes.

Now follows a well-arranged treatment of cyclic collineations. A number of statements are not literally true; *e. g.*, that a cyclic homology of period greater than 2 can not exist (page 162). We gradually see that the author is considering only real elements, but in other articles imaginaries are brought in without warning. The development reminds one of that of Ameseder, but his name is not included among the authors cited. Since the guiding idea is the discussion of forms having two degrees of freedom, quadric and cubic surfaces, space cubic curves and Hirst's complex are legitimate topics of discussion. The polar theory of quadrics is presupposed and a large number of theorems are given without proof, with frequent footnotes giving references to the original memoirs.

Thus far the development has been systematic and comprehensive, and a diligent student could master much of the subject without a teacher, although proofs are much more condensed and the amount of presupposed knowledge much higher than in the preceding volume. The second chapter however (pages 219–346) is concerned almost entirely with problems of enumeration, and is to be regarded as a hand-book, rather than a text. It begins with a short discussion of singular correlations, and uses the results persistently throughout the chapter.

The procedure is as follows: Given the point plane $\Sigma \equiv (A, B, \cdots)$ and the line plane $\Sigma' \equiv (a', b', \cdots)$ correlated in such a way that A has a' for image, B has b', etc., and that B, C, D are collinear, but a', c', d' form a triangle.

The pencil $A(B, C, D)$ and the range $a'(b', c', d')$ are projective, so that corresponding to any line AX will be a definite point X' on a', but other pencils, *e. g.*, $B(A, C, D)$ go into the single point $a'b'$. We can now construct the image of any point or line in either plane. The lines a' and $a \equiv (B, C, D)$ are both singular. This correlation is called axial. If the relation between Σ, Σ' be such that a', c', d' are concurrent, but B, C, D not collinear (dual of the preceding case) the central correlation with the singular points A, A' results.

There are ∞^8 correlations in the plane. If a given point P is associated with a given line p', this counts for two conditions, since two parameters are necessary to fix P. If, however, P is to be on some given line p, without more definitely fixing it, the fact that p, p' are conjugate counts for one condition. Now consider α pairs (p', P), β pairs (P', p), γ pairs (A, A'), δ pairs (a, a') given, wherein

$$2\alpha + 2\beta + \gamma + \delta = 8.$$

The number of polar fields fixed by these conditions is indicated by $(\alpha\beta\gamma\delta)_8 = (\beta\alpha\gamma\delta)_8 = (\alpha\beta\delta\gamma)_8 = (\beta\alpha\delta\gamma)_8$, since α, β and γ, δ can be interchanged by duality.

If now one condition be suppressed, in $(\alpha\beta\gamma\delta)_7$ there are ∞^1 correlations, including a finite number π central collineations, λ axial correlations, μ non-singular ones in which a given pair of points are conjugate, and ν in which two given lines are conjugate. To obtain relations among these numbers, consider two lines a, b given in Σ, and a point C' in Σ'. A line x'_a through C' and a are therefore conjugate in ν correlations. In each of

these construct the pole of b, and connect it with C' by means of x'; these lines are respectively conjugate to b. Thus a (ν, ν) correspondence between the lines of C' is established, having 2ν coincidences made up of π central collineations and μ having a, b and c' conjugate points, hence $2\nu = \mu + \pi$; and by duality $2\mu = \nu + \lambda$. If the number of correlations μ in the pencil having a given pair of conjugate points be denoted by $\mu(\alpha, \beta, \gamma, \delta)_7$, we may therefore write

$$\mu(\alpha, \beta, \gamma, \delta)_7 = (\alpha, \beta, \gamma + 1, \delta)_8.$$

Finally, if two conditions be suppressed, in $(\alpha, \beta\, \gamma, \delta)_8$ are ∞^1 axial and ∞^1 central collineations. The conjugate points will describe curves, the conjugate lines will envelop others. By use of the chapter on multiple correspondence developed in the first volume it is now possible to determine $(\alpha, \beta, \gamma, \delta)_8$ in each case.

Many of the details are omitted; in order to follow the argument in the later cases of the bundle frequent use must be made of Schubert's Enumerative geometry, Hirst's article in the *Mathematische Annalen*, volume 8, and in particular to the comprehensive memoir of the author in volume 12 of the *Mathematische Annalen*. Contrary to the promise in the preface of the first volume, the treatment is synthetic throughout the volume, thus making the development quite one-sided and restricted.

VIRGIL SNYDER.

Arithmétique Graphique. Les Espaces Arithmétiques, leurs Transformations. Par G. ARNOUX. Paris, Gauthier-Villars, 1908. xii + 84 pp.

THIS book treats of arithmetic spaces which have been defined and studied by the author in two previous books written under the same general title, "Arithmétique Graphique."* The second of these was reviewed by the writer in May, 1907.† The reader is referred to this review for the definition of arithmetic spaces and for a short account of Arnoux's use of them to furnish a graphical representation in the theory of numbers.

In this third book, the author is more interested in the properties of the arithmetic spaces themselves and not so much

* Les espaces arithmétiques hypermagiques, Paris, 1894. Introduction a l'étude des fonctions arithmétiques, Paris, 1906.

† BULLETIN, vol. 13, pp. 402–403.

in their applications. He has in mind the creation of what he calls a géométrie analytique arithmétique which shall be to the theory of numbers what ordinary analytic geometry is to algebra and analysis. He does not claim to have made more than a beginning in the working out of his idea, but clearly states that he has tried only to point out the possibility and the usefulness of such an attainment.

In the book under review the author uses coordinates more extensively than in his earlier work. He finds that the general idea of geometric transformation may be conveniently applied to arithmetic spaces, and that the general linear transformation in a two-dimensional arithmetic space changes arithmetic lines into arithmetic lines; that is, it is a collineation. This is true whether the arithmetic space be infinite or modular. In an infinite arithmetic space, the domain for the coordinates of points is the totality of integers, but in an arithmetic space of modulus m the domain consists of integers, modulo m. If m is a prime number, these integers form a field or finite algebra, and consequently the analytic geometry of the space is more like ordinary analytic geometry. In particular, the transformation called inversion is possible. Arnoux calls attention to this fact.

The reviewer takes this opportunity to mention the fact that Arnoux's k-dimensional arithmetic space of prime modulus p is what Veblen and Bussey have called a finite euclidean geometry and have denoted by the symbol $EG[k, p]$.* Arnoux has not given the theorem that the linear transformation in a k-dimensional modular arithmetic space ($k > 2$) of prime modulus is a collineation, nor has he made any study of groups of transformations in these finite euclidean geometries.

The book closes with a chapter of applications to number theory, magic squares, and Euler's problem of the 36 officers.

W. H. BUSSEY.

Fragen der Elementargeometrie, gesammelt und zusammengestellt von FEDERIGO ENRIQUES. Deutsche Ausgabe von DR. HERMANN FLEISCHER. II Teil. pp. xii + 348. Leipzig, B. G. Teubner, 1907.

ONE of the few unique books of the last few years on elementary mathematics was the collection of related monographs

* "Finite projective geometries," *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 241–259. In particular, see §7.

on "Questioni riguardanti la geometria elementare" which appeared in 1900 under the editorship of F. Enriques. The volume under review is more than a translation, it is a thorough revision of the portion of the original that deals with the theory of geometrical construction. The other volume will contain a discussion of postulates, definitions, theory of area, the parallel axiom, and related topics.

In every respect the German edition is an improvement on the original. The text is less involved, and assumes little more than should be acquired in an ordinary college course in the calculus. Even this degree of proficiency is unnecessary for most of the book.

The first chapter contains a discussion of the various methods of attacking original problems of construction, and follows rather closely the excellent little book of Petersen on this subject. The next ninety pages are devoted to constructions that can be performed by the use of particular instruments, as, for example, the compasses alone, where the spirit of Mascheroni's work is very clearly presented; the ruler alone, showing the nature of the projective properties of figures; the parallel ruler; and various instruments in connection with a fixed curve. It is almost a pity that Lill's very elegant constructions of cubic and higher irrationalities by means of the movable right angle are not mentioned.

The analytic side of the subject is not neglected, and a clear demonstration of the impossibility of making constructions of higher order than the second with ruler and compasses is presented. The construction of the 17-gon is, perhaps, given more prominence than is necessary, but this is offset by the concise discussion of the various exact and approximate solutions of the classical problems of trisecting the angle, and duplicating the cube. Several approximate constructions are given for π, and the transcendence of π is demonstrated.

The book is a most welcome contribution to the literature that should be of interest to teachers in secondary schools. It makes no pretense of showing how to teach, but gives a background of fact and theory that no teacher should lack. The writer uses it as required reading in an undergraduate course for prospective teachers, and believes that most of the book can be appreciated by undergraduates. Though the style and aim of most of the chapters is for reading by professional teachers rather than professional mathematicians, at the same time the

number of important ideas to which the reader is introduced is surprisingly large, and what is of as great importance, a breadth of outlook pervades the volume that teachers will certainly appreciate.

H. E. HAWKES.

Leçons de Mécanique céleste professeés à la Sorbonne. Par H. POINCARÉ. Paris, Gauthier-Villars. Tome I, 1905, 365 pp.; tome II, 1re partie, 1907, 165 pp.

TISSERAND has written an excellent four volume treatise on celestial mechanics, containing most of the classic contributions to the subject in the spirit and often in the notations of their original authors. Poincaré's Les Méthodes nouvelles de la Mécanique céleste was devoted to establishing, with the rigor of modern mathematical methods, the existence of various kinds of periodic orbits, to determining their properties, to proving the non-existence of new uniform integrals analytic in the masses and the existence of asymptotic solutions, etc. These profound and important researches have almost nothing in common with earlier investigations either in method or subject matter. The Leçons of Poincaré occupy ground between that covered in Tisserand's treatise and in Les Méthodes nouvelles. They are largely expositions of certain parts of standard theories, but the methods of development are those which the author considers as best suited to reaching the desired end regardless of what other writers may have used. It is doubtful if Poincaré could have compelled himself to take the little steps his predecessors have sometimes taken, or to follow their often circuitous paths. At any rate it is fortunate for us that we are able to see how the most penetrating and brilliant mathematician who has written on celestial mechanics reacts on some of our standard theories.

The first volume of the Leçons is devoted to the general theory of planetary perturbations. The first chapter contains an account of canonical equations, their properties, and their transformations. The canonical equations are used throughout the work. The problem of elliptic motion and the equations for the variations of the elliptic elements are developed in these variables, and Lagrange's method of variation of parameters is explained and applied. It is easily shown that the general term in the expression for either a coordinate or an element is of the form

$$\mu^{\alpha} A M t^{m} \cos(\nu t + h),$$

where μ is a small quantity of the order of the masses, A a function of the ratios of the major semiaxes of the orbits, M a power series in the eccentricities and inclinations, and ν a linear homogeneous function of the mean motions. If $m = 0$, $\nu \neq 0$ the term is *periodic*, if $m \neq 0$, $\nu = 0$ the term is purely *secular*, and if $m \neq 0$, $\nu \neq 0$ the term is *mixed secular*. Poincaré calls α the *order* of the term, the degree of the term of lowest degree in the small quantities in M the *degree* of the term, $\alpha - m$ the *rank* of the term, and, if m' represents the exponent of the small divisor arising through integration, $\alpha - m/2 - m'/2$ the *class* of the term. For practical work the importance of a term depends upon its order, degree, rank, and class, and the classical results relating to the stability of the system are attached to relations among these properties. In Chapter V Poincaré has proved in a very simple way, considering the complexity of the subject, that (1) there are no terms of negative rank; (2) there is no mixed secular term of rank zero; and (3) in the expressions for the major axes there is no term of rank zero. These theorems have been known since the time of Poisson for the first two approximations ($\alpha = 1, 2$), and they are recalled here because Poincaré shows they are general.

Only a few of the many results and methods of interest can even be mentioned here, but the conclusions reached in Chapters VIII and IX cannot be passed in silence. These chapters are devoted to a discussion of the terms of rank zero. It is clear that while these terms may be of little importance for small values of t, they are important in determining the configuration of the system for large values of t, certainly if the series in which they occur are convergent. The original differential equations may be written in the form

$$\frac{dx_i}{dt} = \sum_{j=0}^{\infty} B_{ij} \cos(\nu_{ij} t + h_{ij}).$$

Certain of the ν_{ij} are zero. Let the values of B_{ij} for which this is true be $B_{ij}^{(0)}$, and those for which $\nu_{ij} \neq 0$ be $B_{ij}^{(1)}$. Then we have

$$\frac{dx_i}{dt} = \sum B_{ij}^{(0)} \cos h_{ij} + \sum B_{ij}^{(1)} \cos(\nu_{ij} t + h_{ij}).$$

The first terms in the right members give rise to terms of rank zero and order one. Let the terms in $B_{ij}^{(0)}$ of degree one be

represented by $B_{ij}^{(00)}$. Lagrange treated the question of stability by taking out these terms and writing

$$\frac{dx_i^{(00)}}{dt} = \sum B_{ij}^{(00)} \cos h_{ij}.$$

Since there is no term in the major axes of rank zero, they are constant so far as these equations are concerned. Making use of this fact, the remaining equations become linear and homogeneous with constant coefficients. Lagrange showed that all the roots of the characteristic equation of this system are purely imaginary, and inferred from this result that the solar system is stable. However, if the process of Lagrange be extended to the terms in $B_{ij}^{(0)}$ of higher degree it is found that secular terms appear in the expressions of the third order.

The result obtained by Poincaré, and given also in Les Méthodes nouvelles, volume II, chapter X, is that the equations

$$\frac{dx_i^{(0)}}{dt} = \sum B_{ij}^{(0)} \cos h_{ij}$$

can be transformed and integrated *formally* so that the $x_i^{(0)}$ involve only trigonometric terms.

In chapter X he shows that the general problem of three bodies may be *formally* integrated in purely trigonometric series, a result established long ago by Newcomb and Tisserand.

Volume II, Part 1, of the Leçons is entirely devoted to the development of the perturbative function. The whole problem is treated with the skill of a master. From the point of view of novelty and mathematical interest chapter XX is the most important. It is devoted to a discussion of the convergence of the series for the coefficients of the development of the perturbative function. The problem is the development of a certain function $F(u, u')$, which is periodic in both u and u' considered separately, in a series of the form

$$F = \sum B_{mm'} e^{i(mu+m'u')}.$$

The $B_{mm'}$ are functions of various of the elements and are expanded as power series in certain of these parameters. They are defined by

$$B_{mm'} = \frac{1}{4\pi^2} \int_{|u|=1} \int_{|u'|=1} F e^{-i(mu+m'u')}\, du du'.$$

The singularities of $B_{mm'}$ as functions of the parameters (let us call them $\beta_1, \cdots, \beta_n$) are found by varying $\beta_1, \cdots, \beta_n$ in F, and finding under what conditions the contours of integration can not be deformed so as to escape passing through a singularity. By a very beautiful discussion the general problem is set up, and the details are worked out in some of the simpler cases.

In conclusion, we may inquire to what class of readers these volumes will most appeal. They certainly can not be digested by American students who are just starting work in the lines to which they are devoted. While Poincaré begins at first principles and works out the results in rather complete detail, the giant strides of the master cross too wide a field of mathematics to be within the grasp of beginners in graduate work. The mathematics would all be readily intelligible to the mature mathematician, but those who were not already somewhat familiar with the subject matter would very often find themselves at a loss to know why the numerous particular artifices were employed. For one having an interest in, and knowledge of, celestial mechanics, this work will be a great source of information and inspiration. In these respects it is surpassed only by the incomparable Méthodes nouvelles, which will, I believe, indirectly revolutionize celestial mechanics even in all its practical details.

F. R. MOULTON.

Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte. Gesamtbericht, herausgegeben von A. GUTZMER. Leipzig, 1908, large 8°, pp. xii + 322.

IN a stately volume there is laid before us the complete report of the Commission appointed in 1904 by the Society of German natural scientists and physicians to examine and report upon various proposed reforms in the teaching of mathematics and and the natural sciences in Germany. The present report contains reprints of the proceedings in Cassel (1903), and in Breslau (1904), which led to the formation of the Commission, and of the proposals submitted by the Commission in Meran (1905), Stuttgart (1906), and Dresden (1907).

The central questions relative to the teaching of mathematics were taken up in the proposals of 1905, which have already been discussed in the BULLETIN * and the later reports deal

* Vol. 12 (1906), pp. 347–352.

with matters of smaller general interest or importance. The proposals of 1906 have been reviewed in the BULLETIN;* those of 1907, so far as mathematics is concerned, deal mainly with the scientific preparation of candidates for appointment as teachers. An interesting explanatory address by Professor F. Klein introduces this part of the report, which among many things of a local character contains others of a far wider range of interest. Its spirit cannot be laid before American readers more concisely and effectively than by citing a few characteristic recommendations of wider applicability, whose basic principles, translated if need be into our terminology, deserve as thoughtful consideration in America as in Germany.

It will be recalled that the preparation of the prospective teacher of mathematics in Prussia consists of two parts, the scientific preparation, attained in a minimum of three years of university study (beginning with analytic geometry and the calculus), and the pedagogic preparation acquired by two years of training in a secondary school. Only the scientific preparation is considered in the report, which points out (pages 270, 273) that when once the function concept has been made the center of the mathematics of the schools, as recommended by the Commission in 1905, the introductory university work will be a natural outgrowth and continuation of the secondary work and not separated from it by a marked gap as has hitherto too generally been the case.

It is recommended that the university student should from first to last not only hear lectures but also be constantly employed in active work himself, ranging from the treatment of small problems to independent investigations, reporting thereon before his fellows in seminars. The professors should have assistants in conducting this work (in the schools of technology in Prussia it is already customary to appoint one assistant per 30 students) and suitable libraries and working-rooms are also indispensable.

"The commission also recommends emphatically that at the end of the general studies in pure mathematics a course be given organizing the entire mathematical material according to its essential interrelations and as far as possible presenting the import of the higher branches for the different stages of school mathematics. For, in fact, experience teaches that without such a course of study, the majority of the students do not dis-

* Vol. 13 (1907), pp. 304–305.

cover the inner bond that connects the various parts of mathematical science, and thus the prospective teacher loses what should be for him the real gist of his mathematical studies. To avoid misunderstanding, we add expressly that this course presupposes matured hearers, and should not be brought down to the level of those preparing to teach mathematics as a minor subject only."

The commission also proposes the following tentative scheme of general professional studies for prospective teachers of mathematics and physics. A common program is proposed for the minimum period of three years, specialization in one direction or the other being contemplated in case the student remains at the university longer than three years, as many do. The courses are presumably of four hours each weekly, supplemented throughout by exercises and seminars as recommended above. Little stress is laid upon the order of the courses, which must be varied to correspond with the courses offered by universities that are not able to offer all the courses of the outline each year.

Half Year.	Professional Studies.			General Studies.
First.	Calculus I.	Plane Analytics.	Experimental Physics I.	
Second.	Calculus II.	Descriptive Geometry (with projective geometry).	Experimental Physics II.	
Third.	Differential Equations.	Elementary Mechanics with graphic and numerical methods.	Introduction to Chemistry.	
Fourth.	Algebra with Theory of Numbers.	Curves and Surfaces.	Higher Mechanics.	History of Philosophy and Pedagogy.
Fifth.	Theory of Functions.	Geodesy with Probabilities.	Theoretic Physics I.	Logic.
Sixth.	Synoptic Course.	Astronomy with Geophysics.	Theoretic Physics II.	Psychology.

The report closes with a bibliography (pages 309–322) of 290 titles of publications appearing in the years 1900–07, inclusive, relative to the field of the commission's labors and classified as shown in the table on the next page.

Referring again to the earlier review in the BULLETIN of the important general proposals of the commission, it may be said in closing that this complete report presents in convenient

	General.	Mathematics.	Physics.	Chemistry, Geology, Biology.	Totals.
Books	11	16	5	6	38
Articles:					
Secondary Schools	61	79	14	40	194
Universities	14	28	7	9	58
Totals	86	123	26	55	290

form for permanent preservation and use the record of the entire activity of the commission, an activity which will not lose its significance for many years to come, and whose record will long remain a pedagogic document of the first importance.

J. W. A. YOUNG.

NOTES.

ALL the papers read at the joint meetings of mathematicians and engineers, held at Chicago in December, 1907, have been published in *Science*, and reprinted in the form of a 56-page pamphlet, for the use of the joint committee of fifteen, of which Professor E. V. HUNTINGTON is chairman, and which is to present its report, on the teaching of mathematics in colleges of engineering, at the summer meeting, 1909, of the Society for the promotion of engineering education. A limited number of these reprints are available for distribution among members of the AMERICAN MATHEMATICAL SOCIETY, on application to the Secretary of the Chicago Section, Professor H. E. SLAUGHT, 58th Street and Ellis Avenue, Chicago, Ill. They may be had in the order of application while they last.

THE opening (January) number of volume 10 of the *Transactions of the American Mathematical Society* contains the following papers: "Zur Differentialgeometrie der analytischen Curven," by E. STUDY; "The central of a group," by G. A. MILLER; "The hypergeometric functions of n variables," by J. I. HUTCHINSON; "Surfaces derived from the cubic variety having nine double points in four dimensional space," by V. SNYDER; "On a certain class of isothermic surfaces," by A. E. YOUNG; "A geometrical application of binary syzygies," by A. E. LANDRY; "Definite forms in a finite field," by L. E. DICKSON.

THE January number (volume 10, number 2) of the *Annals of Mathematics* contains: "Concerning a compound discontinuous solution of the surface of revolution of minimum area," by MARY E. SINCLAIR; "On the geometric properties of quartic curves possessing fourfold symmetry with respect to a point," by R. D. CARMICHAEL; "Extension of the sieve of Eratosthenes to arithmetical progressions and application," by J. C. MOREHEAD.

THE American association for the advancement of science will not hold a summer meeting this year, in order to avoid interference with the meeting of the British association at Winnipeg. The winter meeting will be held in Boston under the presidency of President D. S. JORDAN, of Stanford University. Professor E. W. BROWN is vice-president and Professor G. A. MILLER secretary of Section A.

THE second regular meeting of the Syracuse Section of the teachers of mathematics in the middle states and Maryland was held at the Syracuse high school, December 29, 1908. The following papers were presented. By H. DE W. GROAT, "Field notes on the teaching of secondary mathematics"; by A. M. CURTIS and C. E. BIKLÉ, "What should be the aims in teaching algebra, and how to attain them?"; by W. E. BOND and E. P. SISSON, "What should be the aims in teaching geometry and how to attain them?"

THE committee on organization of the International commission on the teaching of mathematics has appointed as the members of the commission to represent the United States Professors W. F. OSGOOD, D. E. SMITH, and J. W. A. YOUNG. The preliminary report, setting forth the general plan of the work, was published in *L'Enseignement Mathématique*, November 15, 1908.

FOLLOWING the amendment to the constitution enacted in August, 1907, the council of the Circolo matematico di Palermo now consists of five resident and forty non-resident members. Among the latter are Professor E. H. MOORE, of the University of Chicago, and Professor W. F. OSGOOD, of Harvard University.

AT the annual public meeting of the Paris academy of sciences, held on December 7, 1908, the following prizes were

awarded for memoirs in pure and applied mathematics: Grand prize (fr. 3000) divided equally between Professor L. BIANCHI, of the University of Pisa, and Professor C. GUICHARD of the University of Clermont-Ferrand, for their memoirs on the deformations of quadric surfaces; Francœur prize (fr. 1000) to Dr. E. LEMOINE, for his contributions to geometry; Poncelet prize (fr. 2000) to Professor I. FREDHOLM, of the University of Stockholm, for his researches in functional analysis; Montyon prize (fr. 700) to E. LEBERT, chief engineer of the department of roads and bridges, for his contributions to mechanics; Fourneyron prize, not awarded; Guzman prize not awarded; Lalande prize (fr. 540) divided between Director W. L. ELKIN, of the Cape Town Observatory, and Mr. F. L. CHASE, of Yale University, with honorable mention of Mr. M. F. SMITH, of Yale University, for their investigations of stellar parallax; Valz prize (fr. 460) to Professor M. LUIZET, of the University of Lyons, for his researches in astronomy; Janssen prize (fr. 400) to Professor P. PUISEUX, of the University of Paris, Wilde prizes (two, fr. 2000 each) to Dr. M. TIKHOFF, of the Observatory of St. Petersburg, and Dr. C. NORDMANN, of the Observatory of Paris, for their researches on the dispersion of light and on variable stars; Houllevigue prize (fr. 500) was divided between Professor DEBIERNE, of the University of Paris, Professor PETOT, of the University of Lille, and Professor E. FABRY, of the University of Montpellier; Delcros prize (fr. 8000) to Professor J. HADAMARD, of the University of Paris; Laplace prize (complete works of Laplace) to Dr. P. M. E. LANCRENON, of the national school of mines.

The following programme of proposed prize subjects was announced. Grand prize (fr. 3000) (1910): "It is known how to find all the systems of two meromorphic functions of one complex variable, connected by an algebraic relation. An analogous problem is proposed for three uniform functions of two complex variables, having for finite regions the character of rational functions, and connected by an algebraic relation. In case a complete solution is not obtained, the academy demands at least some illustrative examples which lead to classes of new transcendental functions;" Francœur prize (fr. 1000) (1909) for work of merit in pure or applied mathematics; Poncelet prize (fr. 2000) in 1909 for a meritorious memoir in applied mathematics, in 1910 for a similar contribution to pure mathematics; Bordin prize (fr. 3000) (1911). "Complete in some impor-

tant point, the theory of systems of triply orthogonal surfaces;" Montyon prize (fr. 700) (1910): "Invent or perfect useful machines in agriculture or the mechanic arts;" Vaillant prize (fr. 4000) (1911): "Complete, in some important point, the theory of the movement of an ellipsoid in an infinite, viscous liquid." Besides these, a number of prizes are offered for meritorious advances in any science.

THE philosophical faculty of the University of Göttingen announces the following problem, for the solution of which the faculty will award the Beneke prize in 1911:

"The number of oscillations which have been observed in the spectra of the elements seem to be distributed according to some definite law. They form so-called series. All the known observations should be collected and studied, and theories deduced from them should be critically examined. Moreover, further experiments are desired, to complete the known observations. It is probable that barium will show a three-fold series, corresponding to the known three-fold series of related substances."

Competing memoirs should be sent to the secretary before August 31, 1910. The result of the competition will be announced March 11, 1911. The value of the first prize is 3400 Marks, of the second is 680 Marks.

THE curators of the Wolfskehl foundation announce that Professor H. POINCARÉ has consented to deliver at Göttingen, April 22–28, a series of lectures on various mathematical subjects. Mathematicians generally are invited to attend these lectures. Sessions of the Göttingen mathematical society have also been arranged for the same week.

THE publishing house of B. G. Teubner in Leipzig and Berlin announces that the following mathematical works are in the press and will probably appear within a few weeks: "Encyklopädie," parts of volumes II_2, III_1, III_2, IV_2, V_2, V_3, VI_1, VI_2; "Encyclopédie," all the remaining parts of tome I, volumes 1, 2, 3, 4; part of tome II, volume 1, tome IV, volumes 2 and 3; "Grundlagen der Geometrie" (3d edition), by D. HILBERT; "Elementar-Mathematik vom höheren Standpunkte aus: Teil 2, Geometrie," by F. KLEIN; "Handbuch der Mathematik," by F. MEYER, H. THIEME, E. NETTO, and C. FÄRBER; "Ein-

führung in die höhere Algebra," by M. BÔCHER; "Grundzüge der Differential- und Integralrechnung," by G. KOWALEWSKI; "Vorlesungen über Variationsrechnung" (part 2), by O. BOLZA; "Projektive Geometrie der Ebene," by H. GRASSMANN; "Die Lehre von den geometrischen Verwandtschaften, Band III, Die eindeutigen linearen Verwandtschaften zwischen Gebilden dritter Stufe," by R. STURM. The second installment of volume 1 in the second edition of Clebsch-Lindemann's "Vorlesungen über Geometrie," will also soon appear. The plan for the compilation of the second edition of volume 2 of Pascal's "Repertorium der höheren Mathematik" has now been completed. It is hoped that the German edition will appear before the end of the present year.

PROFESSOR M. DISTELI, of the technical school at Dresden, has accepted a professorship of mathematics at the technical school of Karlsruhe.

PROFESSOR A. SOMMERFELD, of the University of Munich, has been elected associate member of the Bavarian academy of sciences.

PROFESSOR LIAPOUNOFF, of the University of St. Petersburg, has been elected corresponding member of the Academia dei Lincei at Rome.

DR. L. KOLLROS has been appointed docent in mathematics at the technical school of Zürich.

DR. H. HERTZER, emeritus professor of mathematics at the technical school of Charlottenberg, died November 16, 1908, at the age of 78 years.

DR. W. F. WHITE, head of the department of mathematics in the State Normal School at New Paltz, N. Y., died November 29, 1908.

SECOND-HAND catalogue: H. Hugendubel, 18 Salvatorstrasse Munich. Catalogue No. 39, containing 603 mathematical titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARNDT (B.). Ueber die Verallgemeinerung des Krümmungsbegriffes für Raumkurven. (Diss.) Königsberg, 1908. 8vo. 117 pp.

BRANDES (H.). Ueber die axiomatische Einfachheit mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes. (Diss.) Halle, 1908. 8vo. 50 pp.

BURKHARDT (H.). Funktionentheoretische Vorlesungen. Vol. I. 1tes Heft: Algebraische Analysis. 2te, durchgesehene und vermehrte Auflage. Leipzig, Veit, 1908. 8vo. 12 + 199 pp. Cloth. M. 6.60

BUZENGEIGER (K.). See FEUERBACH (K. W.).

CARUS (P.). The foundations of mathematics: a contribution to the philosophy of geometry. Chicago, Open Court Pub. Co., 1908. 12mo. 4 + 141 pp. Cloth. $0.75

DARBOUX (G.). See FABRY (E.).

DIECK (W.). Zur Klassifikation der Punktepaar- und Kegelschnitt-Büschel. (Diss.) Kiel, 1908. 8vo. 40 pp.

ERFURTH (P.). Die Komplementärflächen der pseudosphärischen Schraubenflächen. (Diss.) Halle, 1908. 8vo. 59 pp.

FABRY (E.). Traité de mathématiques générales à l'usage des chimistes, physiciens, ingénieurs et des élèves des facultés des sciences. Avec une préface de G. Darboux. Paris, Hermann, 1909. 8vo. 10 + 440 pp. Fr. 9.00

FEUERBACH (K. W.). Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks und mehrerer durch sie bestimmten Linien und Figuren. Eine analytisch-trigonometrische Abhandlung. Mit einer Vorrede von K. Buzengeiger. 2te nichtgeänderte Ausgabe. Haarlem, 1908. 8vo. 13 + 73 + 6 pp. M. 2.25

FILON (L. N. G.). An introduction to projective geometry. London, Arnold, 1908. 8vo. 258 pp. Cloth. 7s. 6d.

HEILAND (F.). Hüllflächen einer Schar von Regelflächen zweiter Ordnung. (Diss.) Jena, 1908. 8vo. 69 pp.

HEINEMANN (E.). Die gemischte Kegelschnittschar [Das Kegelschnittsystem S (1 p, 3 l)]. (Diss.) Giessen, 1908. 8vo. 84 pp.

HELLINGER (E.). See KLEIN (F.).

HENSEL (K.). Theorie der algebraischen Zahlen. (In 2 Bänden.) Vol. I. Leipzig, Teubner, 1908. 8vo. 9 + 349 pp. Cloth. M. 14.00

KELLER (S. S.) and KNOX (W. F.). Mathematics for engineering students: Analytic geometry and calculus. 2d edition revised. New York, Van Nostrand, 1908. 8vo. 356 pp. Cloth. $2.00

KLEIN (F.). Elementarmathematik vom höheren Standpunkte aus. 1ter Teil: Arithmetik, Algebra, Analysis. Vorlesung, gehalten im Wintersemester 1907–08. Ausgearbeitet von E. Hellinger. Leipzig, Teubner, 1908. 8vo. 8 + 590 pp. (Lithogr.) M. 7.50

KNOX (W. F.). See KELLER (S. S.).

LEVY-BRUHL (L.). See RUSSELL (B.).

METZ (J. E. VON). Beweis des Fermat'schen Satzes. Göttingen, Spielmeyer, 1908. 8vo. 6 pp. M. 0.20

NICHOLS (E. W.). Analytic geometry for colleges, universities and technical schools. Revised edition. Boston, Heath, 1908. 12mo. 11 + 282 pp. Cloth. $1.25

NIELSEN (N.). Lehrbuch der unendlichen Reihen. Vorlesung, gehalten an der Universität Copenhagen. Leipzig, Teubner, 1909. 8vo. 8 + 287 pp. M. 12.00

PASCAL (E.). Esercizi critici di calcolo differenziale e integrale. 2a edizione riveduta. Milano, Hoepli, 1909. 16mo. 15 + 275 pp.

POINCARÉ (H.). L'invention mathématique. Conférence faite à l'Institut général psychologique. Paris, 1908. 8vo. 15 pp.

RAY (J. et R. J.). See RUSSELL (B.).

REYE (T.). Die Geometrie der Lage. Vorträge. 5te, verbesserte und vermehrte Auflage. Leipzig, Kröner, 1909. 8vo. 8 + 255 pp. M. 10.00

ROTHENBERG (S.). Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen totaler Differentialgleichungen von der ersten Ordnung mit zwei variablen Grössen. (Diss.) München, 1908. 8vo. 86 pp.

RUSSELL (B.). La philosophie de Leibniz. Exposé critique. Traduit de l'anglais par J. Ray et R. J. Ray. Avec une préface de l'auteur et un avant propos par L. Levy-Bruhl. Paris, Alcan, 1908. Fr. 3.75

SANIELEVICI (S.). Sur les équations différentielles des cordes et des membranes vibrantes. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 81 pp.

SCHNEE (W.). Ueber irreguläre Potenzreihen und Dirichletsche Reihen. 1ter Teil. (Diss.) Berlin, 1908. 8vo. 80 pp.

TENZ (O.). Ueber singuläre Asymptotenkurven. (Diss.) Giessen, 1908. 8vo. 35 pp.

UMFAHRER (J.). Beweis der Richtigkeit des grossen Fermatschen Satzes. München, Scholl, 1908. 8vo. 10 pp. M. 0.40

WEYL (H.). Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems. (Diss.) Göttingen, 1908. 8vo. 86 pp.

ZÖLLICH (H.). Beiträge zur Theorie der ganzen transzendenten Funktionen der Ordnung Null. (Diss.) Halle, 1908. 8vo. 58 pp.

II. ELEMENTARY MATHEMATICS.

ARMISTEAD (H.). Practical and theoretical geometry for schools. Complete. London, Longmans, 1908. 8vo. 278 pp. Cloth. 2s. 6d.

ARNDT (E.). See MINK (W.).

BASSI (A.). Esercizi e problemi di algebra complementare, ad uso del secondo biennio degli istituti tecnici. Vol. I (per la 3a classe), parte I. Bra, Racca, 1909. 8vo. 143 pp. L. 1.20

BERSANO (G. B.). Algebra per la terza classe delle scuole tecniche e per la prima classe delle scuole normali. Torino, 1908. 8vo. 106 pp. L. 1.60

BIFFIS (P.). Geometria elementare, per le scuole tecniche e normali, con numerosi problemi ed esercizi. Bergamo, Bolis, 1908. 16 mo. 262 pp. L. 2.00

BOUDIER (A.). Algèbre élémentaire (classe de mathématiques). Dijon, Berthier, 1908. 8vo. 82 pp.

——. Notions d'algèbre pratiques (classes de 4e, 3e, 2e A, 1re A). Dijon, Berthier, 1908. 8vo. 16 pp.

CIAMBERLINI (C.). Elementi di geometria, per le scuole tecniche. 2a edizione riveduta e corretta. Torino, Paravia, 1909. 8vo. 136 pp. L. 2.00

DIEKMANN. See KOPPE.

FAIFOFER (A.). Elementi di algebra, ad uso degl'istituti tecnici e dei licei. 17a edizione. Venezia, Sorteni, 1908. 8vo. 437 pp. L. 3.00

——. Elementi di trigonometria piana, e tavole logaritmico-trigonometriche, ad uso dei licei. 27a edizione. Venezia, Sorteni, 1908. 8vo. 92 + 68 pp. L. 2.00

——. Trattato d'aritmetica pratica e nozioni elementari di geometria, ad uso del ginnasio inferiore. 4a edizione. Venezia, Sorteni, 1908. 8vo. 59 pp. L. 2.00

——. Trattato di aritmetica teorico-pratica e principii d'algebra, ad uso delle scuole tecniche e normali. 27a edizione. Venezia, Sorteni, 1908. 8vo. 200 pp. L. 2.00

——. Trattato di geometria intuitiva, ad uso delle scuole tecniche e normali. 39a edizione. Venezia, Sorteni, 1908. 8vo. 165 pp. L. 2.00

FRENCH (C. H.) and OSBORN (G.). Elementary algebra. New York, Putnam, 1908. 8vo. 506 pp. Cloth. $1.50

GHERARDI (U.). Elementi di aritmetica e di geometria, per le scuole tecniche. Vol. II, per la II classe. Milano, Trevisini, 1908. 8vo. 170 pp. L. 1.50

——. Elementi di geometria e algebra, per le scuole techniche. Vol. III, per la III classe. 2a edizione interamente rifatta. Milano, Trevisini, 1908. 8vo. 204 pp. L. 1.75

GUERRERA (V.). La quadratura del circolo, ovvero un nuovo rapporto tra circonferenza e diametro. Catania, Giannotta, 1908. 8vo. 15 pp. L. 3.00

IRIBARREN (L.). Pizarras de geometría con todos los cálculos y figuras, con arreglo á la obra que sirve de texto para el ingreso en las Academias Militares. Toledo, Gómez, 1908. P. 5.00

JACOB (J.) und SCHIFFNER (F.). Lehrbuch der Arithmetik für Unterrealschulen. 2te Abteilung. Lehrstoff der 2ten Klasse. Wien, Deuticke, 1908. 8vo. 58 pp. M. 1.20

JIMÉNEZ RUEDA (C.). Lecciones de geometría métrica, en que se contesta á los programas de las escuelas especiales y de la facultad de ciencias. 2a edicion aumentada. Madrid, Suárez, 1908. 6 + 626 pp. P. 25.00

——. Programa de geometría métrica. Madrid, Suárez, 1908. 23 pp. P. 1.00

KELLER (S. S.). Mathematics for engineering students; Algebra and trigonometry. 2d edition, revised. New York, Van Nostrand, 1908. 8vo. 9 + 282 pp. Cloth. $1.75

——. Mathematics for engineering students: Plane and solid geometry. 2d edition, revised. New York, Van Nostrand, 1908. 8vo. 212 pp. Leather. $1.25

KNOPS (K.). See KOPPE.

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. 20te Auflage. Ausgabe für Reallehranstalten. 2ter Teil der Planimetrie, Stereometrie und Trigonometrie. 4te Auflage der neuen Bearbeitung von K. Knops. Essen, Baedeker, 1908. 8vo. 4 + 250 pp. M. 2.40

LIEBER (H.). See MÜSEBECK (C.).

LÜHMANN (F. VON). See MÜSEBECK (C.).

MARKS (C. I.). Mathematical questions and solutions. New series. Vol. 4. London, Hodgson, 1908. 8vo. 6s. 6d.

MARTIN MENGOD (A.). Elementos de geometría Euclidiana. Orense, 1907. 302 pp.

MATRICULATION mathematics papers, set at the examination of the University of London from Jan. 1900 to Sept. 1908. London, Clive, 1908. 8vo. 132 pp. 1s. 6d.

MINK (W.). Lehrbuch der Geometrie als Leitfaden beim Unterrichte an höheren Lehranstalten. 10te Auflage, bearbeitet von E. Arndt. 2ter Teil. Trigonometrie und Stereometrie. Gotha, 1908. 8vo. 8 + 125 pp. M. 1.80

MÜLLER (H.). Aufgaben zu planimetrischen Konstruktionen und graphischen Darstellungen. Ergänzungsheft zu der Aufgabensammlung von H. Müller und M. Kutnewsky und dem Lehrbuch der Mathematik von H. Müller. Leipzig, Teubner, 1909. 8vo. 6 + 70 pp. M. 0.80

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 10a edizione aumentata delle tavole dei logaritmi d'addizione e sottrazione, per cura di M. Rajna. Milano, Hoepli, 1909. 16mo. 36 + 191 pp.

MÜSEBECK (C.). Aufgaben für den Unterricht in der Goniometrie und ebenen Trigonometrie, enthaltend Aufgaben zu Teil III: Ebene Trigonometrie des Leitfadens der Elementar-Mathematik von H. Lieber und F. von Lühmann. Berlin, Simion, 1908. 8vo. 6 + 78 pp. M. 1.80

NAPRAVNIK (F.). Geometrie und geometrisches Zeichnen für Mädchen-Bürgerschulen. 3 Teile. Wien, Tempsky, 1908. 8vo. 159 pp. M. 2.40

OSBORN (G.). See FRENCH (C. H.).

PATERSON (W. E.). School algebra. Part I. New York, Oxford University Press, 1908. 12mo. 328 pp. Cloth. $0.60

PREDELLA-LONGHI (L.). Elementi di algebra, ad uso della prima classe normale. 5a edizione, riveduta e corretta. Torino, Paravia, 1908. 8vo. 104 pp. L. 1.60

RAJNA (M.). See MÜLLER (O.).

RUEFLI (J.). Elementare Theorie der Maxima und Minima nebst Aufgaben zur Uebung. Bern, Francke, 1908. 8vo. 7 + 79 pp. M. 2.00

SALOMON (A.). Leçons de géométrie à l'usage de l'enseignement secondaire des jeunes filles. Géométrie dans l'espace (classe de 5e année). 3e édition. Paris, Vuibert, 1909. 16mo. 129 pp.

——. Leçons de géométrie à l'usage de l'enseignement secondaire des jeunes filles. Géométrie plane (classes de 3e et 4e années). 4e édition. Paris, Vuibert, 1909. 16mo. 230 pp. Fr. 2.50

SCHIFFNER (F.). See JACOB (J.).

——. Planimetrie (Ebene Geometrie). 2ter Teil. Leitfaden für den Unterricht in der Geometrie und dem geometrischen Zeichnen in der III. Klasse an österreichischen Realschulen. Mit einem Anhang über das geometrische Zeichnen. Wien, Deuticke, 1908. 8vo. 53 pp. M. 1.00

THIELE (W.). Tafel der Wolfram'schen hyperbolischen 48-stelligen Logarithmen. Bearbeitet und erweitert. Neue Ausgabe. Dessau, Dünnhaupt, 1908. 8vo. 118 pp. M. 6.00

TOURS (F. J.). Eléments de géométrie comprenant des notions sur les courbes usuelles, un complément sur le déplacement des figures et de nombreux exercices. Paris, Poussielgue, 1909. 16mo. 12 + 523 pp.

TURNBULL (V. M.). Cassell's elementary algebra. London, Cassell, 1908. 8vo. 254 pp. Cloth. 2s. 6d.

WOODWARD (C. J.). ABC five-figure logarithms for general use. 2d edition. London, Simpkin, 1908. 12mo. 166 pp. Cloth. 3 s.

WORTNER (F.). Geometrie und geometrisches Zeichnen für Knaben-Bürgerschulen. 3 Teile. 3te Auflage. Wien, Tempsky, 1908. 8vo. 195 pp. M. 3.20

——. Dasselbe. Ausgabe in 1 Rande. M. 2.20

——. Dasselbe für Mädchen-Bürgerschulen. 3 Teile. M. 2.20

III. APPLIED MATHEMATICS.

ALLITSCH (K.). Die Eisenbeton-Konstruktionen. Leitfaden für die Berechnung und Ausführung von einfach verstärkten Betonplatten, Rippendecken und Säulen zusammengestellt für Studierende der Werkmeister-Schule. 1ter Teil: Die Berechnung. Innsbruck, Vereinsbuchhandlung, 1908. 8vo. 63 pp. M. 2.00

ANDRÉE (W. L.). Die Statik des Kranbaues. München, Oldenbourg, 1908. 8vo. 8 + 220 pp. Cloth. M. 8.00

BALL (R. S.). A treatise on spherical astronomy. Cambridge, University Press, 1908. 8vo. 518 pp. Cloth. 12 s.

BENNETT (C. A.). Problems in mechanical drawing. Peoria, Manual Arts Press, 1908. 88 pp. $1.20

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 1re partie à l'usage des élèves des classes de 1re C et D. 6e édition. Paris, Vuibert, 1908. 8vo. 8 + 336 pp. Fr. 3.50

——. Traité de géométrie descriptive. 2e partie, à l'usage des élèves de mathématiques A et B (programme de 1905) et des candidats aux écoles navales, Saint-Cyr, etc. 4e édition. Paris, Vuibert, 1909. 8vo. 272 pp.

COLVIN (F. H.). Machine shop calculations. New York, Hill, 1908. 174 pp. Cloth. $1.00

DIECKMANN (E.). See SCHULTZ (E.).

DUFOUR (F. O.). Bridge engineering; roof trusses; a manual of practical instruction in the calculation and design of steel truss and girder bridges

for railroads and highways, including also the analysis and design of roof-trusses and other details of mill building construction. Chicago, American School of Correspondence, 1908. 8vo. 254 + 96 + 7 pp. Cloth. $3.00

EDSER (E.). Heat for advanced students. New York, Macmillan. 1908. 12mo. 8 + 492 pp. Cloth. $1.00

EXAMPLES of calculating designs. New York, Industrial Press, 1908. 8vo. 48 pp. $0.25

GARNERI (A.). Corso elementare di disegno geometrico. Parte I (Problemi grafici geometrici e ornamentazione geometrica). Ventunesima edizione. Torino, Paravia, 1908. 16mo. 8 + 134 pp. L. 2.00

——. Corso elementare di disegno geometrico. Parte II–III (Proiezioni ortogonali e applicazioni alla meccanica, teoria delle ombre, prospettiva parallela, prospettiva pratica). 6a edizione. Torino, Paravia, 1908. 16mo. 2 vols., pp. 159; 95. L. 3.00

MESSERSCHMITT (J. B.). Die Schwerebestimmung an der Erdoberfläche. (Sammlung naturwissenschaftlicher und mathematischer Monographien, 27tes Heft.) Braunschweig, Vieweg, 1908. 8vo. 8 + 158 pp. M. 5.00

MINEUR (P.). See CHOLLET (T.).

MONTANARI (C.). Elementi di geometria descrittiva, ad uso degli istituti tecnici. 2a edizione riveduta e corretta. Livorno, Giusti, 1909. 16mo. 40 pp. L. 0.40

PALAGI (F.). Nozioni elementari di meccanica, acustica e cosmografia, per la seconda classe dei licei. 3a edizione accresciuta (ristampa). Torino, Loescher, 1908. 8vo. 10 + 254 pp. L. 3.50

SAXELBY (F. M.). An introduction to practical mathematics. London, Longmans, 1908. 8vo. 228 pp. 2s. 6d.

SCHMIEDEL (O.). Die Statik des Eisenbetonbaues. Elementares Lehrbuch zum Gebrauch an Schulen und zum Selbstunterricht. Wien, Kreidel, 1909. 8vo. 5 + 166 pp. M. 3.00

SCHOENFLIES (A.). Einführung in die Hauptgesetze der zeichnerischen Darstellungsmethoden. Leipzig, Teubner, 1908. 8vo. 5 + 92 pp. M. 2.80

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